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Navier-Stokes evolutions as selfdual variational problems

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Abstract

Many nonlinear PDEs and evolutions—including stationary and dynamic Navier-Stokes equations—can be formulated and resolved variationally by minimizing energy functionals of the form

$$I(u) = L(u, -\Lambda u) + \langle \Lambda u, u \rangle$$

and

$$I(u) = \int_0^T \Big[L(t, u(t), -\dot{u}(t) - \Lambda u(t)) + \langle \Lambda u(t), u(t) \rangle \Big] \, dt + \ell(u(0) - u(T), \frac{u(T) + u(0)}{2})$$

where L is a time-dependent "selfdual Lagrangian" on state space, ℓ is another selfdual "boundary Lagrangian", and Λ is a nonlinear operator (such as $\Lambda u = \mathrm{div}(u \otimes u)$ in the NSE case). However, just like the selfdual Yang–Mills equations, the equations are not obtained via Euler–Lagrange theory, but from the fact that a natural infimum is attained. In dimension 2, we recover the well known solutions for the corresponding initial-value problem as well as periodic and anti-periodic ones, while in dimension 3 we get Leray solutions for the initial-value problems, but also solutions satisfying $u(0) = \alpha u(T)$ for any given α in $(-1, 1)$. It is worth noting that our variational principles translate into Leray's energy identity in dimension 2 (resp., inequality in dimension 3). Our approach is quite general and does apply to many other situations.