Fokker–Planck–Kramer equation, hypoellipticity and Witten Laplacians

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Abstract

The lecture will present a summary of works essentially done with F. Hérau and B. Helffer about the question of the return to the equilibrium for the Fokker–Planck–Kramers equation:

$$\left\{\begin{array}{l} \partial_t f + v.\partial_x f - \frac{1}{m}\left(\partial_x V(x)\right).\partial_v f - \frac{\gamma_0}{m\beta}\left(\partial_v - \frac{m\beta}{2}v\right).\left(\partial_v + \frac{m\beta}{2}v\right) f = 0 \\ f(x,v,t=0) = f_0(x,v). \end{array}\right.$$

Take $m = \gamma_0 = \beta = 1$. The non-selfadjoint operator

$$K = \partial_x - \left(\partial_x V(x)\right).\partial_v - \left(\partial_v - \frac{1}{2}v\right).\left(\partial_v + \frac{1}{2}v\right)$$

has to be understood as an hypoelliptic operator for which the natural elliptic operator is the Witten Laplacian on 0-forms

$$\Delta_\Phi^{(0)} = \left[-\Delta_x + \frac{1}{4}\left|\nabla V(x)\right|^2 - \frac{1}{2}\Delta V(x)\right] + \left[-\Delta_v + \frac{v^2}{4} - \frac{d}{2}\right].$$

We are speaking here of global hypoellipticity and one important issue of this analysis is the equivalence

$$\left((1+K)^{-1} \text{ compact}\right) \Leftrightarrow \left((1+\Delta_\Phi^{(0)})^{-1} \text{ compact}\right).$$

This statement can be estasblished under rather general assumptions on the potential $V(x)$ but not all the interesting cases.

It makes use of different techniques for hypoellipticity that we will briefly present here.
Finally we will also present the quantitative results which were obtained in terms of the physical parameters m, γ_0, β.