Elliptic and Hyperelliptic Curve Cryptography

Renate Scheidler

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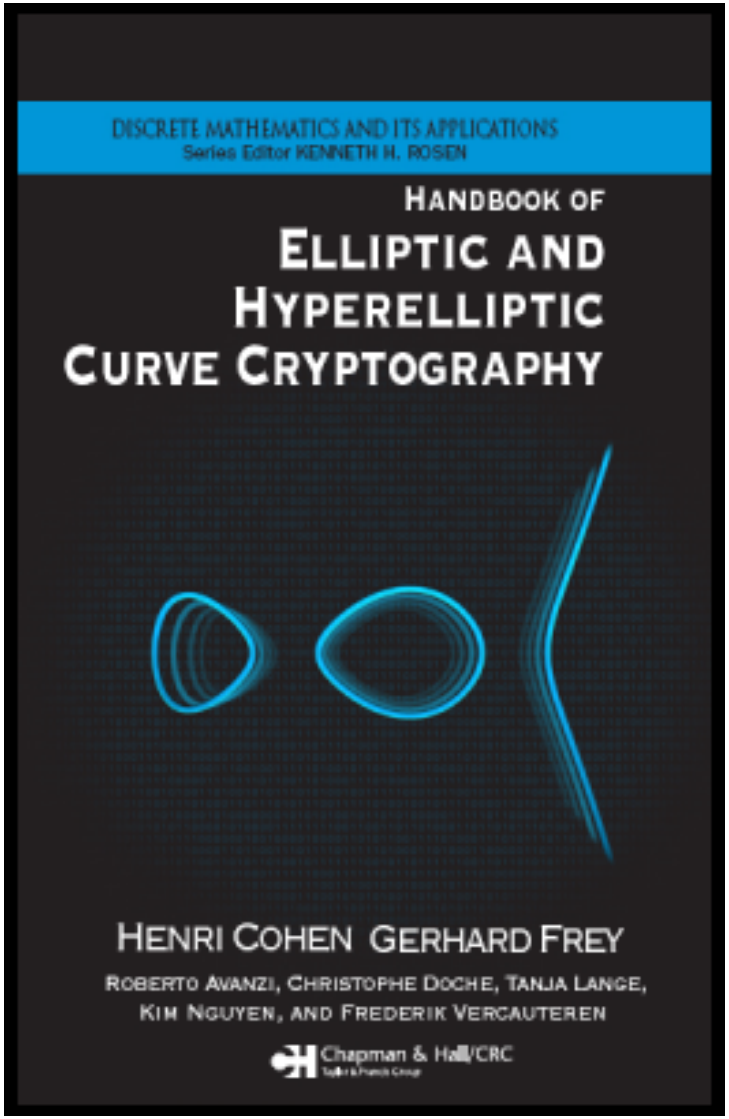
Comprehensive Source

Handbook of Elliptic and Hyperelliptic Curve Cryptography



Overview

- Motivation
- Elliptic Curve Arithmetic
- Hyperelliptic Curve Arithmetic
- Point Counting
- Discrete Logarithm Algorithms
- Other Models

Motivation — Why (Hyper-)Elliptic Cryptography?

Requirements on groups for discrete log based cryptography

- Large group order (plus other restrictions)
- Compact representation of group elements
- Fast group operation
- Hard Diffie-Hellman/discrete logarithm problem

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Elliptic and low genus hyperelliptic curves do well on all of these!

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Let K be a field (in crypto, $K = \mathbb{F}_q$ with q prime or $q = 2^n$)

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Elliptic curve: Weierstraß equation & **non-singularity** condition: there are no simultaneous solutions to $(*)$ and

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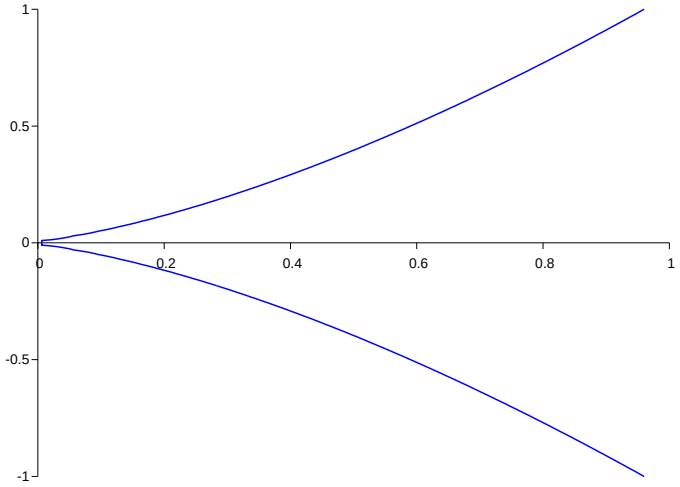
Non-singularity $\iff$ $\Delta \neq 0$ where Δ is the **discriminant** of E

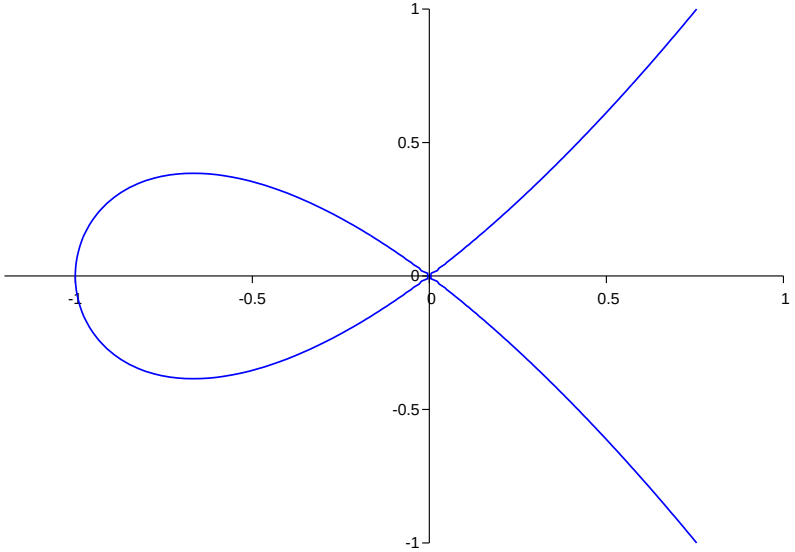
Non-Examples

Two Weierstraß equations with a singularity at $(0,0)$

$$y^2 = x^3$$

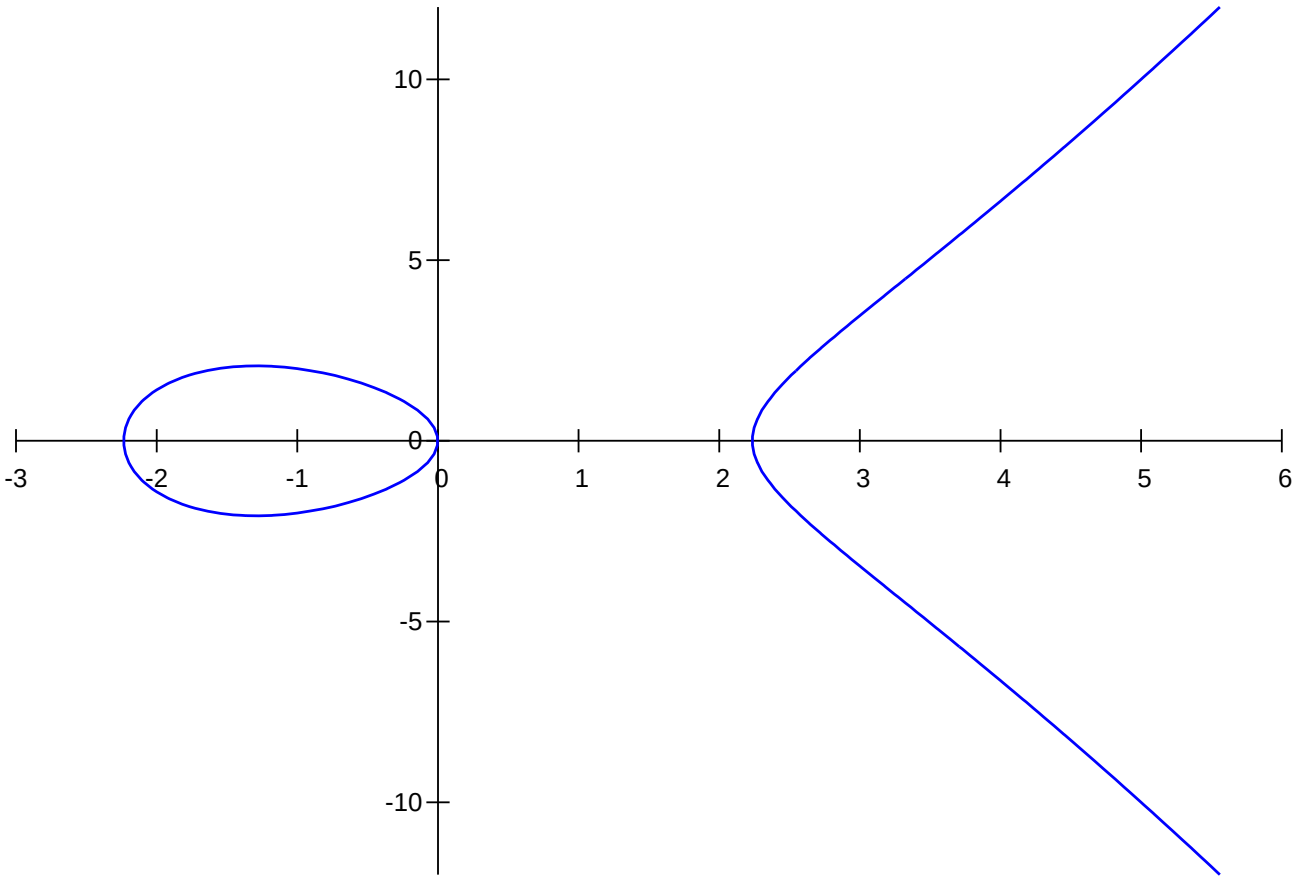


$$y^2 = x^2(x-1)$$



An Example

$E : y^2 = x^3 - 5x$ over $\mathbb{Q}$



Elliptic Curves, $\mathrm{char}(K) \neq 2, 3$

The variable transformations

$$y \to y - (a_1 x + a_3)/2, \text{ then } x \to x - (a_1^2 + 4a_2)/12:$$

yield an elliptic curve in **short** Weierstraß form:

$$E \ : \ \boxed{y^2 = x^3 + Ax + B} \qquad (A, B \in K)$$

Discriminant $\Delta = 4A^3 + 27B^2 \neq 0$ (cubic in x has distinct roots)

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For any field L with $K \subseteq L \subseteq \overline{K}$:

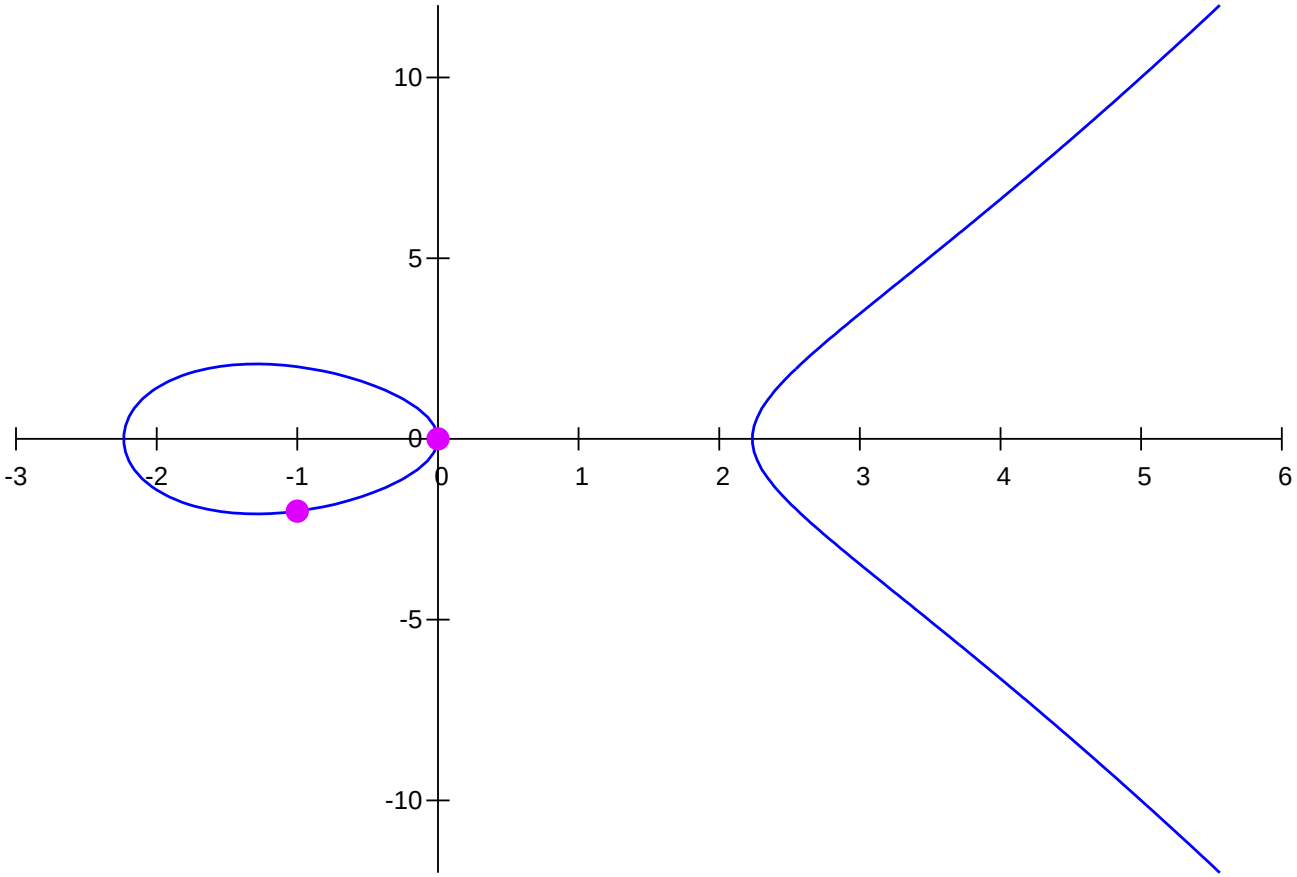
$$\boxed{E(L) \ = \ \{(x_0, y_0) \in L \times L \mid y_0^2 = x_0^3 + Ax_0 + B\} \cup \{\infty\}}$$

set of L-**rational points** on E

An Example

$$E(\mathbb{Q}) = \{(x_0, y_0) \in \mathbb{Q} \times \mathbb{Q} \mid y_0^2 = x_0^3 - 5x_0\} \cup \{\infty\}$$

$$P_1 = (-1, 2),\ P_2 = (0, 0) \ \in \ E(\mathbb{Q})$$



The Mysterious Point at Infinity

In E, replace x by x/z, y by y/z, then multiply by z^3:

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Affine Points		Projective Points
(x, y)	$\leftrightarrow$	$[x : y : 1]$
∞	$\leftrightarrow$	$[0 : 1 : 0]$

Arithmetic on E

Goal: Make $E(L)$ into an additive (Abelian) group

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By **Bezout's Theorem**, any line intersects E in three points

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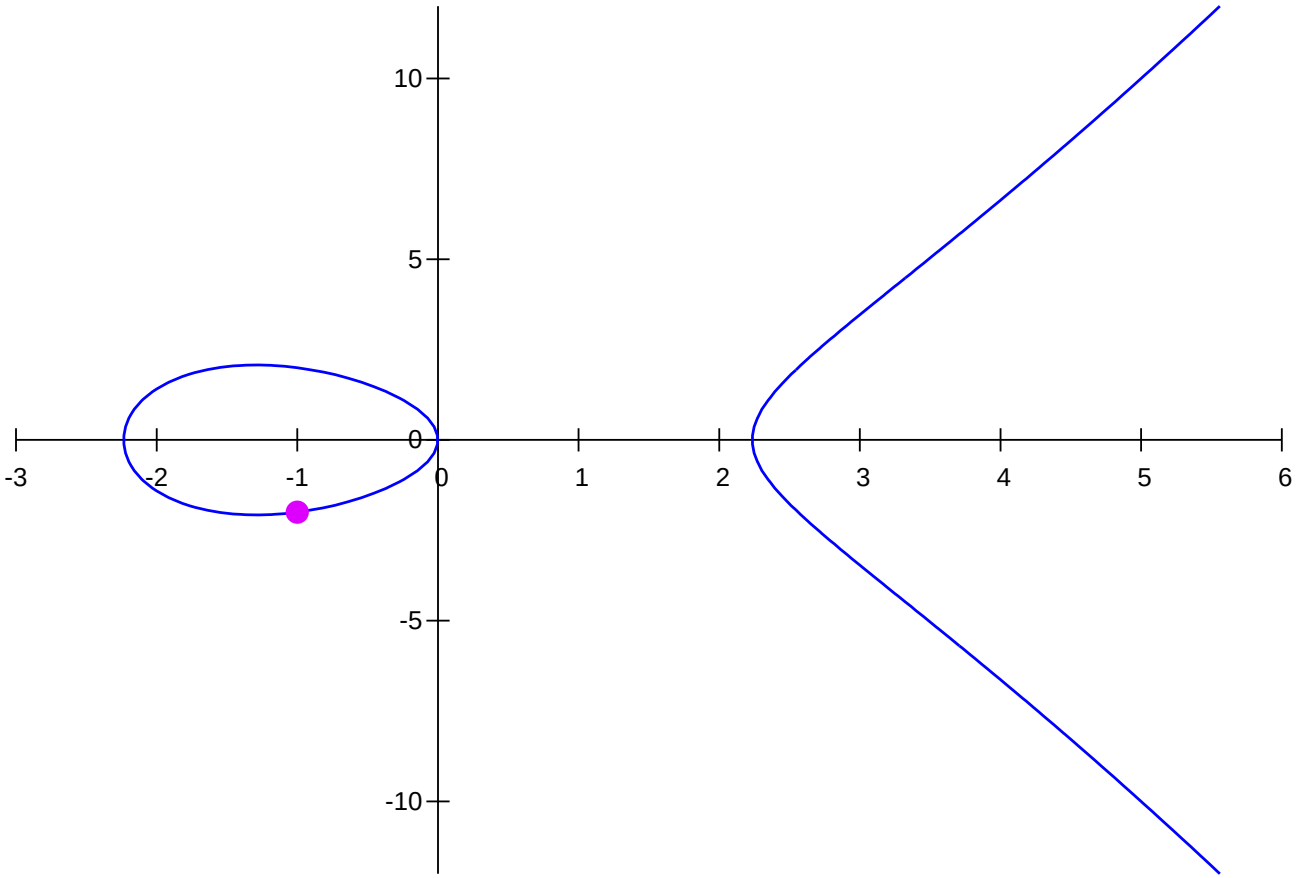
Motto: "Any three collinear points on E sum to zero"

AKA **Chord & Tangent Addition Law**

Arithmetic on E — Inverses

$E : y^2 = x^3 - 5x$ over $\mathbb{Q}$, $P = (-1, -2)$



The line through P and ∞ is $\boxed{x = -1}$

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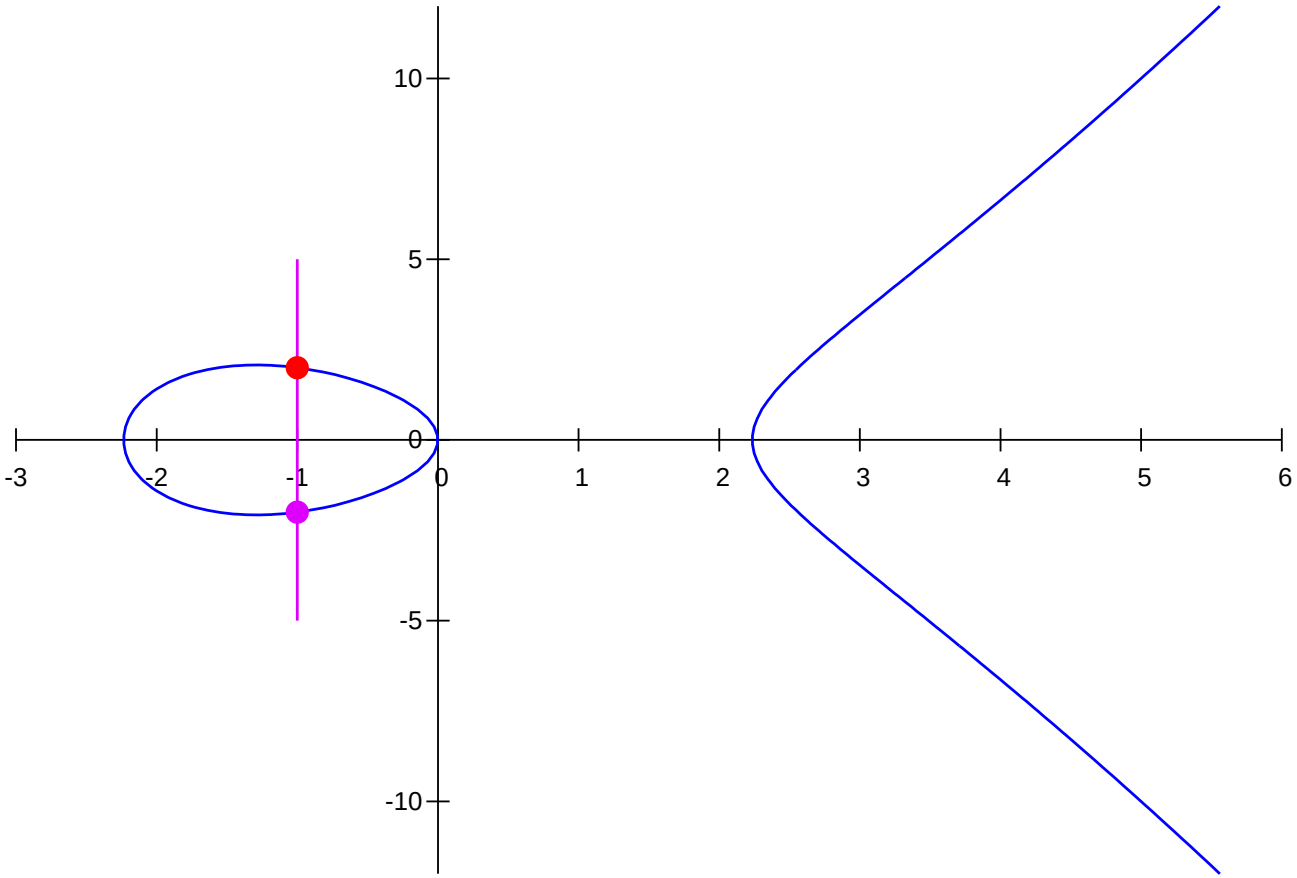
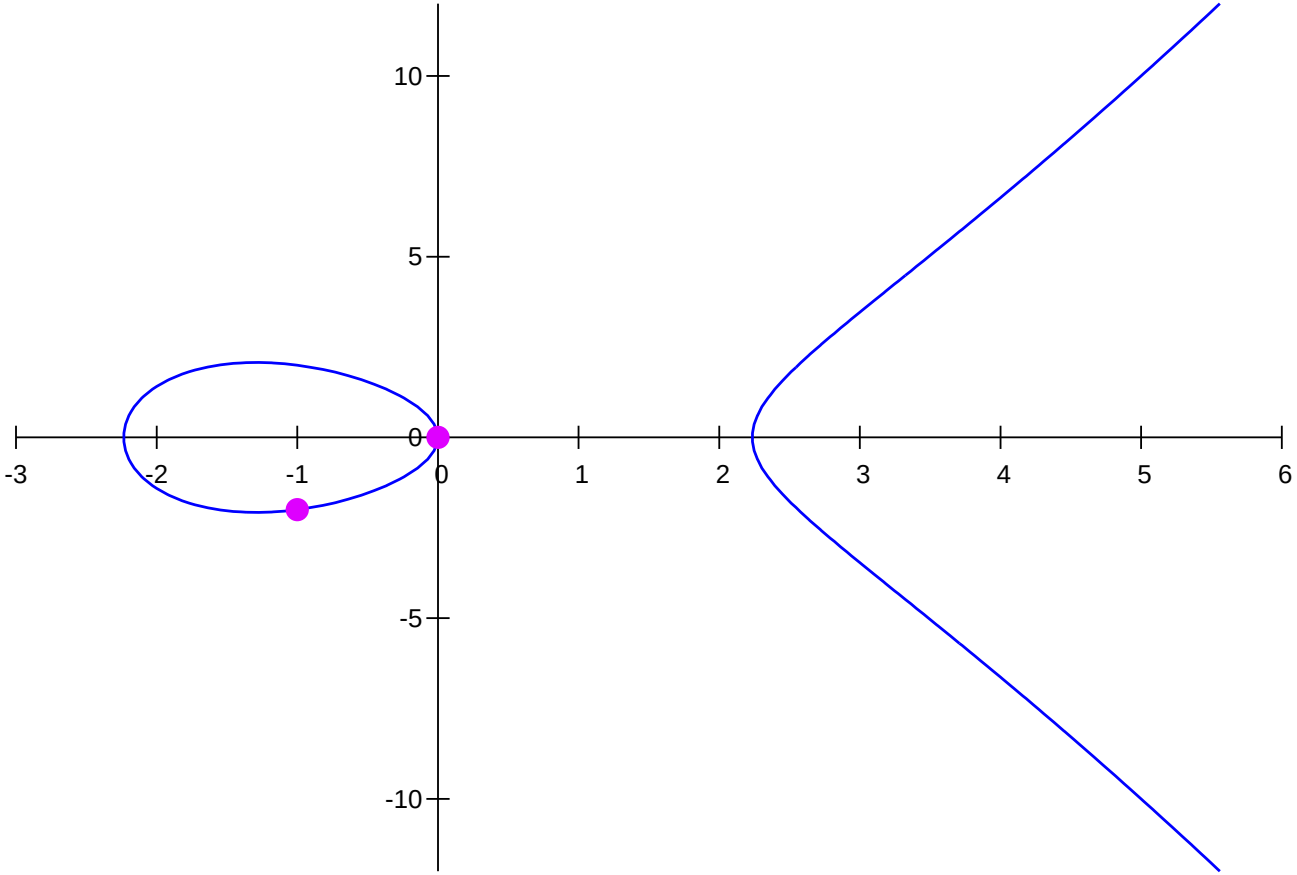


It intersects E in the third point $R = (-1, 2) = -P$

Arithmetic on E — Addition

$E : y^2 = x^3 - 5x$ over $\mathbb{Q}$, $P_1 = (-1, -2)$, $P_2 = (0, 0)$



The line through P_1 and P_2 is $\boxed{y = 2x}$

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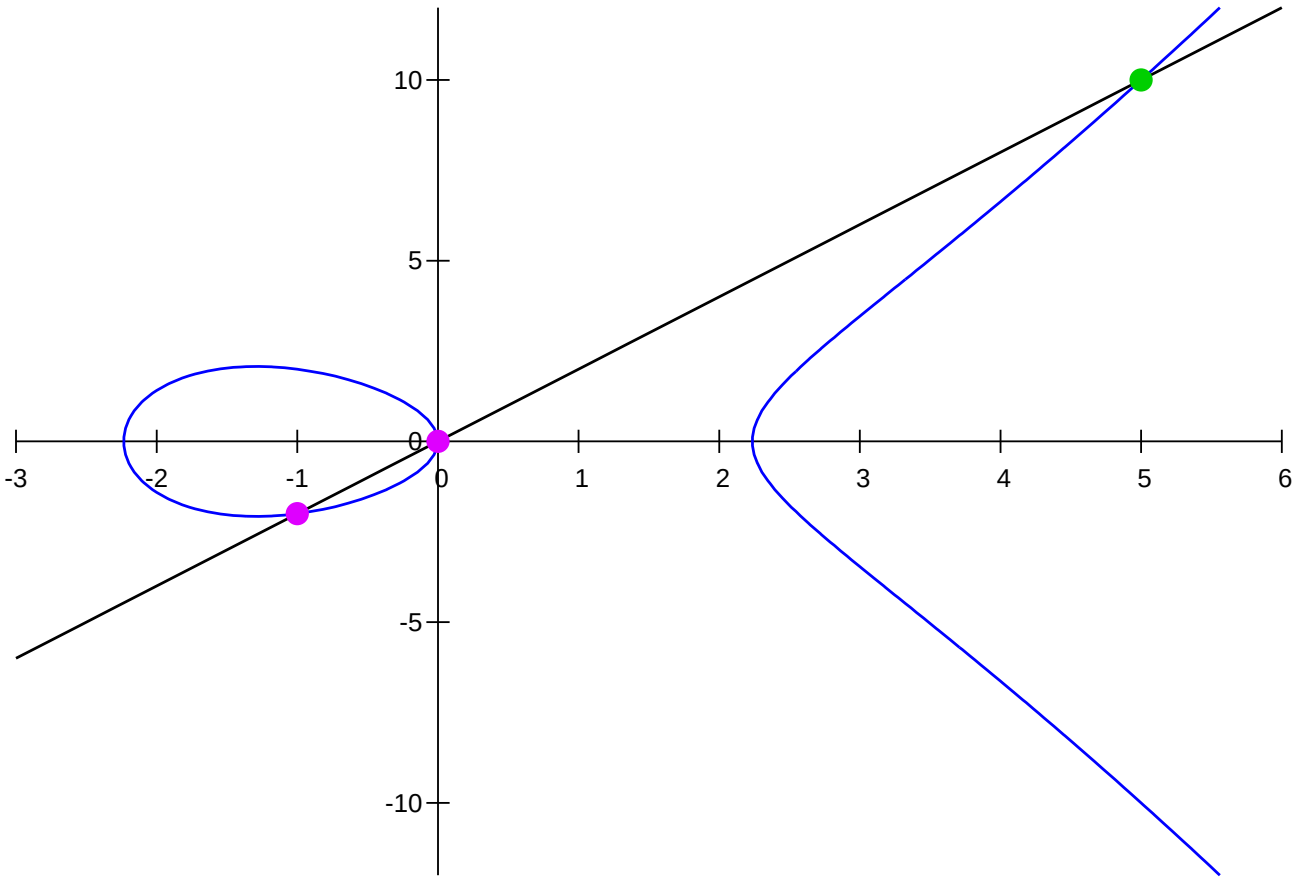

It intersects E in the third point $G = (5, 10)$

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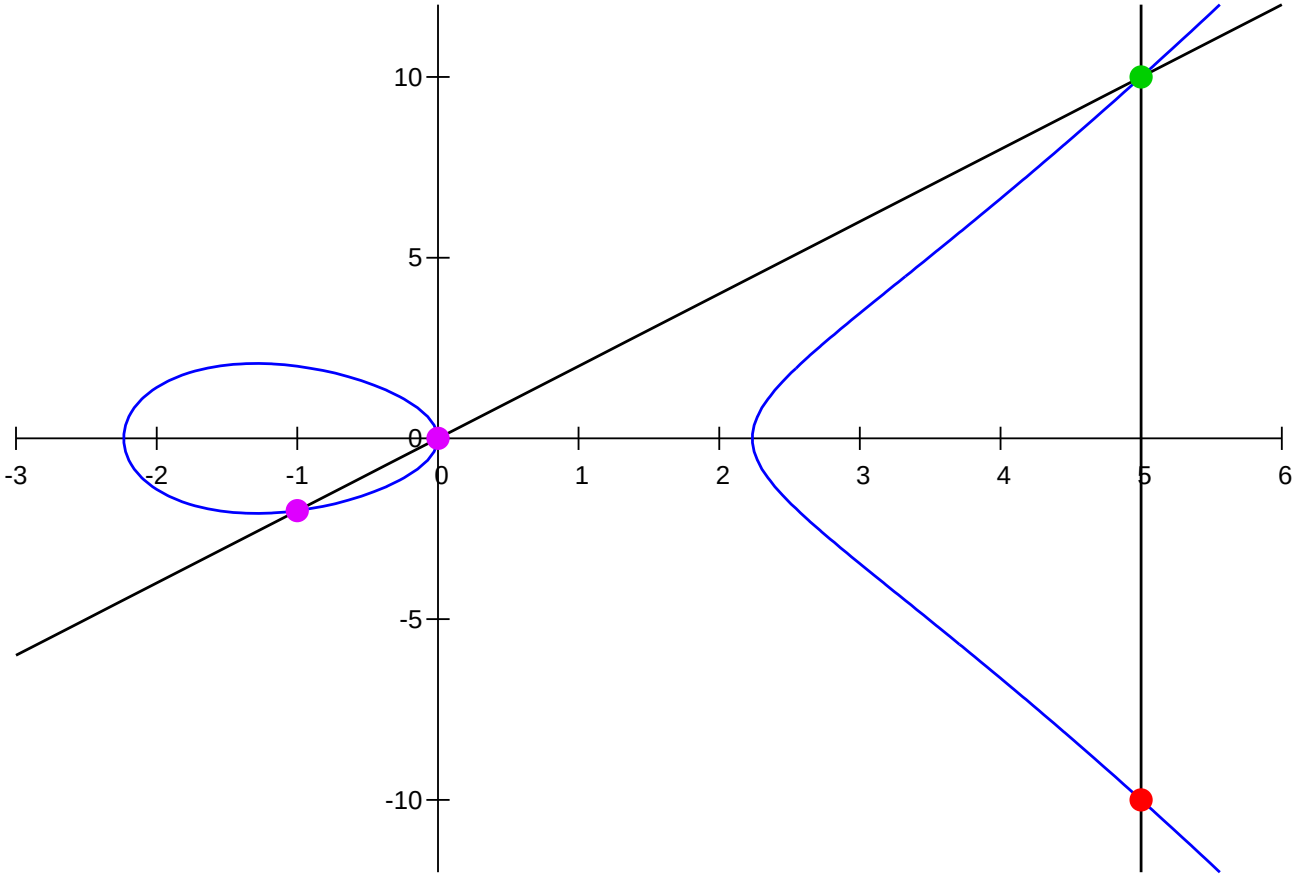

The sum R is the inverse of G, i.e. $R = -G = (5, -10) = P + Q$

Arithmetic on E — Doubling

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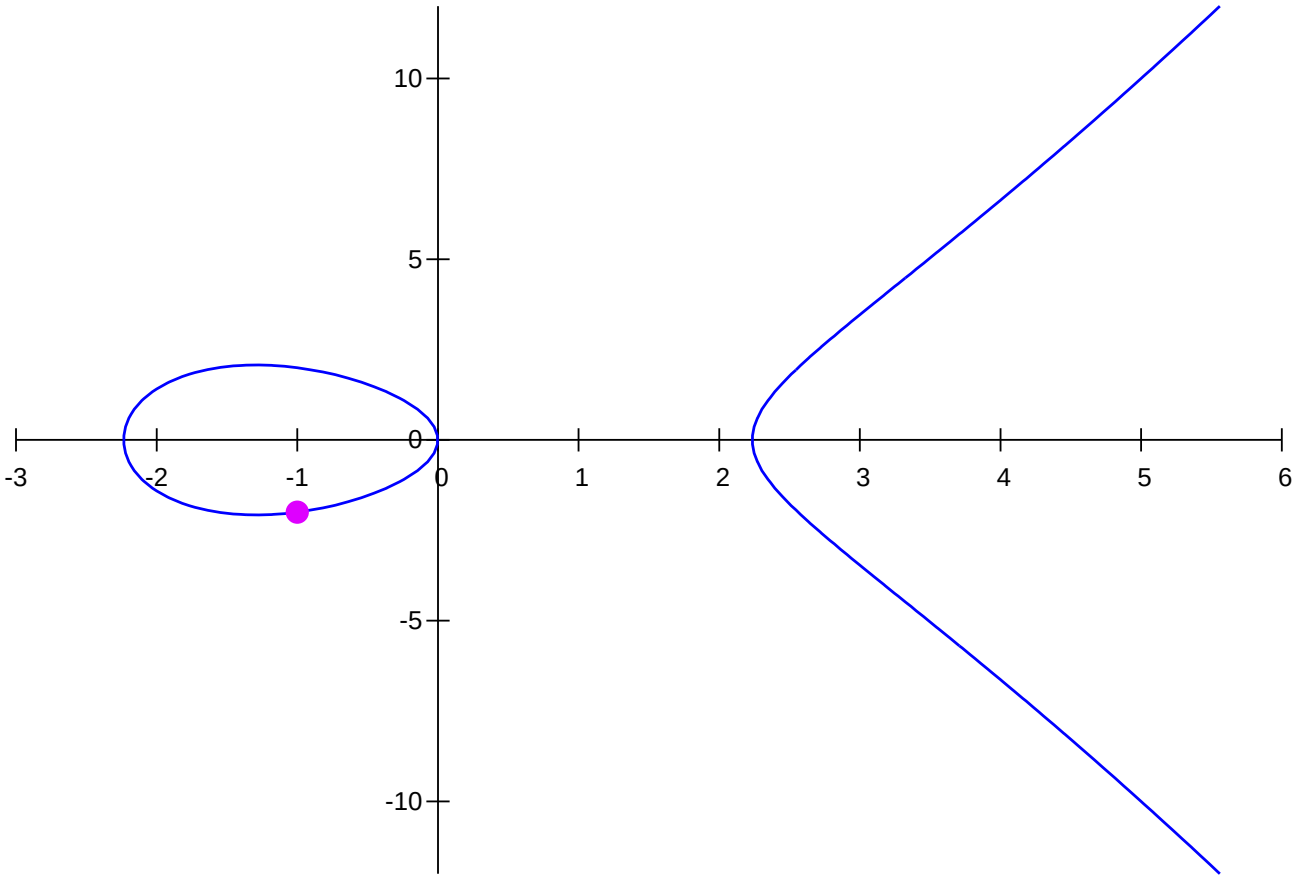

The line tangent to E at P is $\boxed{y = \frac{19}{26}x - \frac{33}{26}}$

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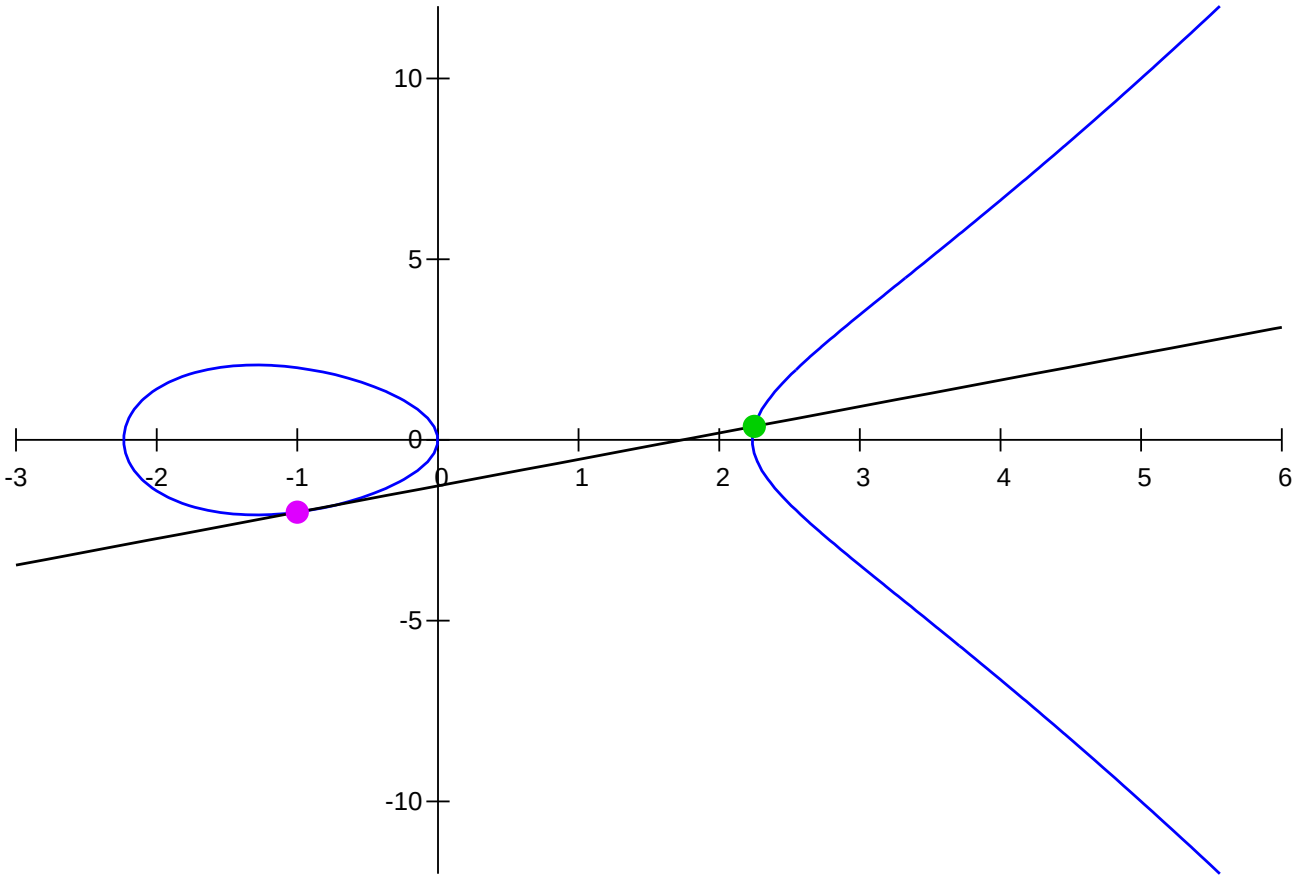

It intersects E in the third point $G = \left(\frac{9}{4}, \frac{3}{8}\right)$

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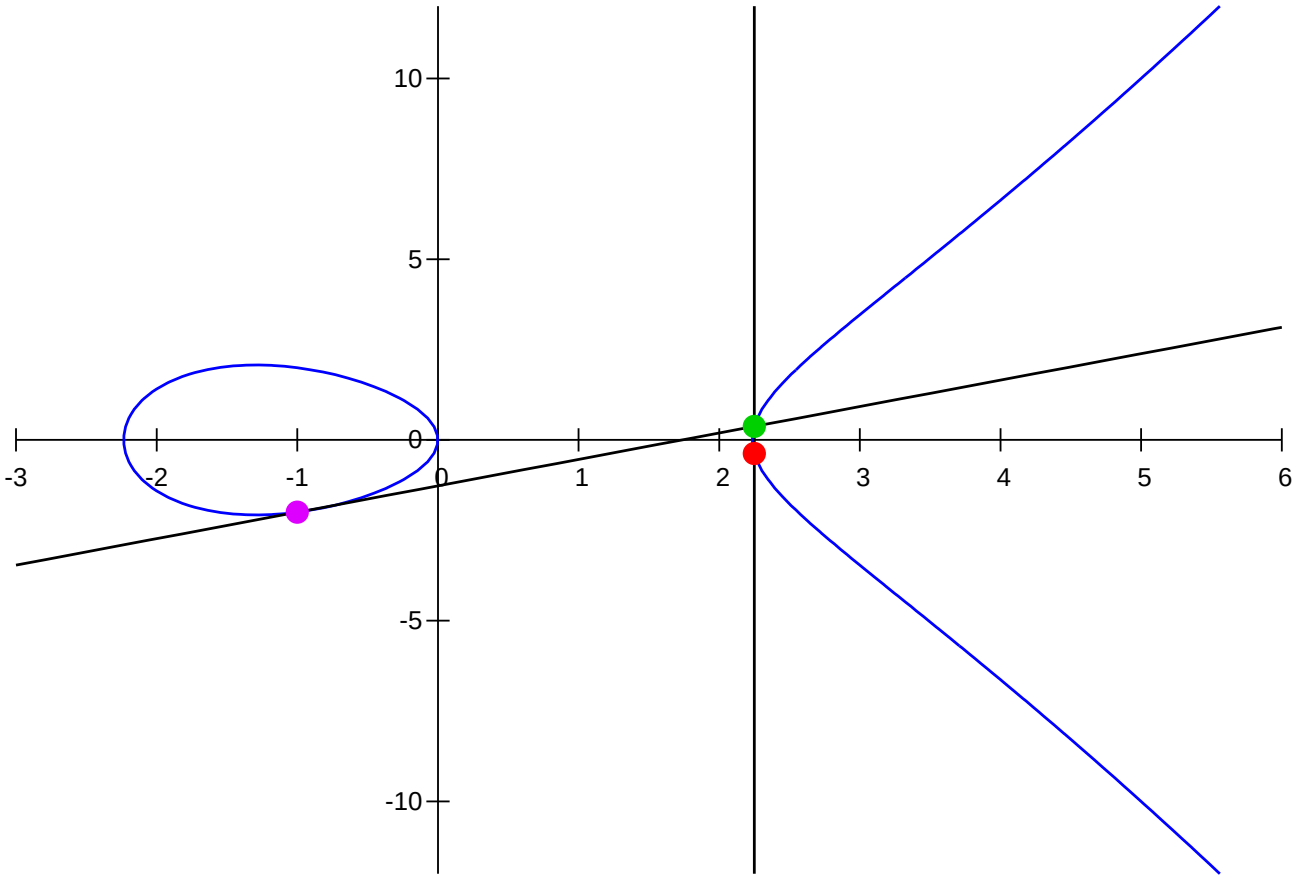

The sum R is the inverse of G, i.e. $R = -G = \left(\frac{9}{4}, -\frac{3}{8}\right) = 2P$

Arithmetic on Short Weierstraß Form — Summary

$$P_1 = (x_1, y_1), \quad P_2 = (x_2, y_2) \qquad (P_1 \neq -P_2;\ P_1, P_2 \neq \infty)$$

$$-P_1 = (-x_1, y_1)$$

$$P_1 + P_2 = (\lambda^2 - x_1 - x_2,\ -\lambda^3 + \lambda(x_1 + x_2) - \mu) \text{ where}$$

$$\lambda = \begin{cases} \dfrac{y_2 - y_1}{x_2 - x_1} & \text{if } P_1 \neq P_2 \\[2ex] \dfrac{3x_1^2 + A}{2y_1} & \text{if } P_1 = P_2 \end{cases}$$

$$\mu = \begin{cases} \dfrac{y_1 x_2 - y_2 x_1}{x_2 - x_1} & \text{if } P_1 \neq P_2 \\[2ex] \dfrac{-x_1^3 + A x_1 + 2B}{2y_1} & \text{if } P_1 = P_2 \end{cases}$$

Beyond Elliptic Curves

Recall Weierstraß equation:

$$E \ : \ y^2 + \underbrace{(a_1 x + a_3)}_{h(x)} y = \underbrace{x^3 + a_2 x^2 + a_4 x + a_6}_{f(x)}$$

$\deg(f) = 3 = 2 \cdot 1 + 1$ odd

$\deg(h) = 1$ for $\mathrm{char}(K) = 2$; $h = 0$ for $\mathrm{char}(K) \neq 2$

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g is the **genus** of the curve

$g = 1$: elliptic curves

$g = 2$: $\deg(f) = 5$, $\deg(h) \leq 2$ — also good for crypto

Hyperelliptic Curves

Hyperelliptic curve of genus g over K:

$$H \ : \ y^2 + h(x)y = f(x)$$

- $h(x), f(x) \in K[x]$
- $f(x)$ monic and $\deg(f) = 2g + 1$ is odd
- $\deg(h) \leq g$ if $\mathrm{char}(K) = 2$; $h(x) = 0$ if $\mathrm{char}(K) \neq 2$
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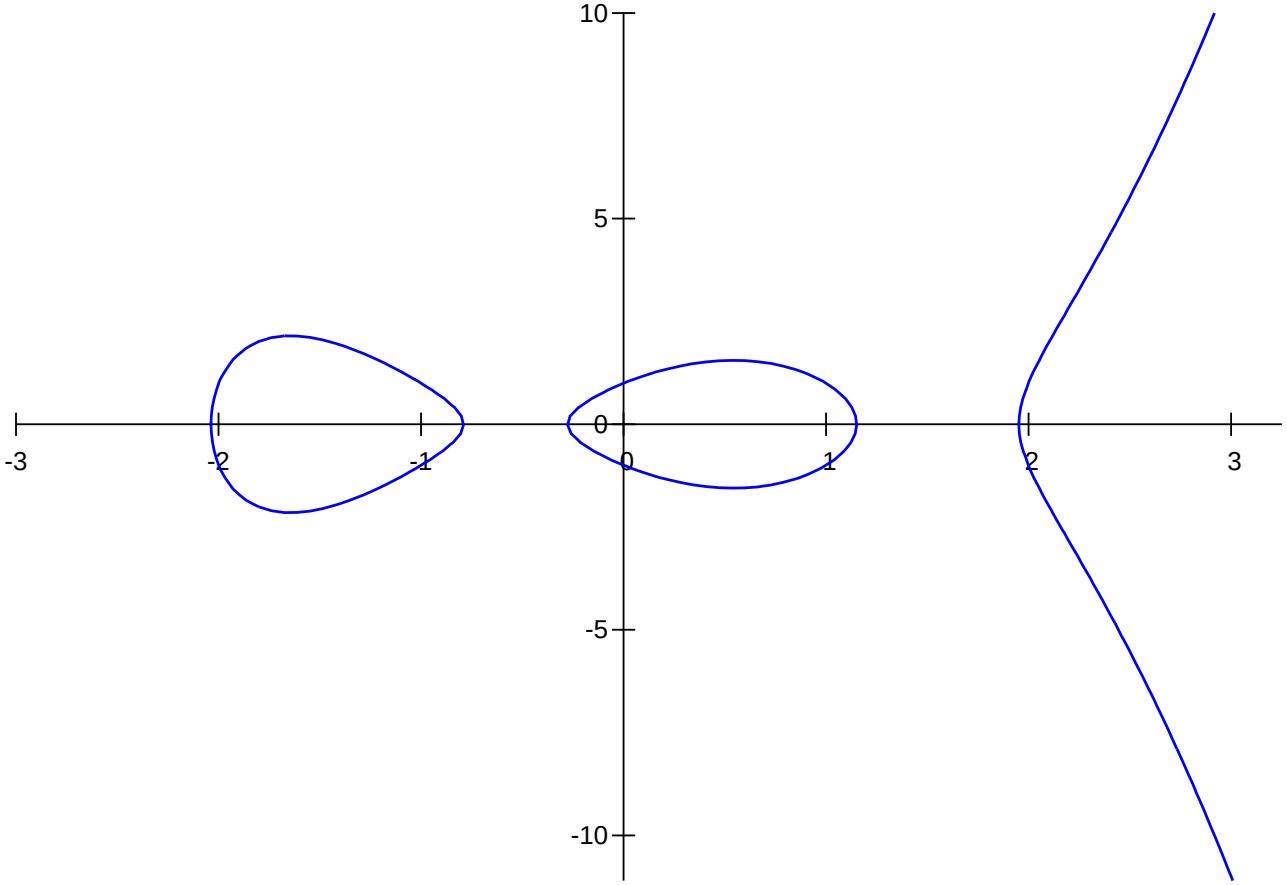
Set of L-**rational** points on H ($K \subseteq L \subseteq \overline{K}$):

$$H(L) \ = \ \{(x_0, y_0) \in L \times L \mid y_0^2 + h(x_0)y = f(x_0)\} \cup \{\infty\}$$

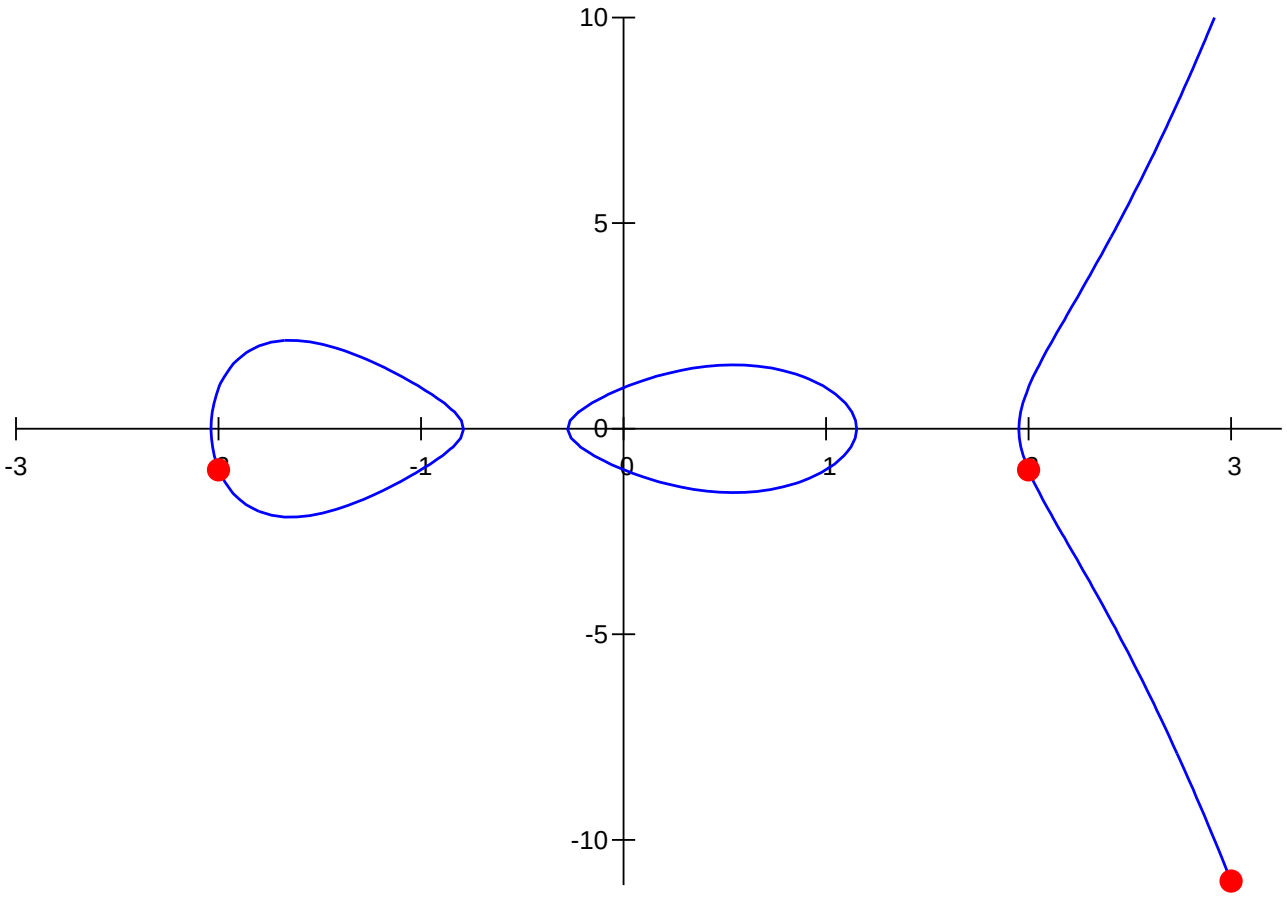
An Example

$H : y^2 = x^5 - 5x^3 + 4x - 1$ over $\mathbb{Q}$, genus $g = 2$



An Example

$(-2,-1),\ (2,-1),\ (3,-11)\ \in\ H(\mathbb{Q})$



Divisors

- Group of **divisors** on H:

$$\mathrm{Div}_H(\overline{K}) = \langle H(\overline{K}) \rangle = \left\{ \sum_{\text{finite}} m_P P \mid m_P \in \mathbb{Z},\ P \in H(\overline{K}) \right\}$$

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$$\mathrm{Div}_H^0(\overline{K}) = \langle [P] \mid P \in H(\overline{K}) \rangle = \left\{ \sum_{\text{finite}} m_P [P] \mid m_P \in \mathbb{Z},\ P \in H(\overline{K}) \right\}$$

where $[P] = P - \infty$

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- Subgroup of $\mathrm{Div}_H^0(\overline{K})$ of **principal divisors** on H:

$$\mathrm{Prin}_H(\overline{K}) = \left\{ \sum_{\text{finite}} v_P(\alpha)[P] \mid \alpha \in K(x,y),\ P \in H(\overline{K}) \right\}$$

The Jacobian

Jacobian of H: $\mathrm{Jac}_H(\overline{K}) = \mathrm{Div}^0_H(\overline{K})/\mathrm{Prin}_H(\overline{K})$

Motto: "Any complete collection of points on a function sums to zero"

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Inverses: The points

$$P = (x_0, y_0) \quad \text{and} \quad \overline{P} = (x_0, -y_0 - h(x_0))$$

on H both lie on the function $x = x_0$, so

$$-[P] = [\overline{P}]$$

Semi-Reduced and Reduced Divisors

Every class in $\mathrm{Jac}_H(\overline{K})$ contains a divisor $\sum_{\text{finite}} m_P[P]$ such that

- all $m_P > 0$ (replace $-[P]$ by $[\overline{P}]$)
- if $P = \overline{P}$, then $m_P = 1$ (as $2[P] = 0$)
- if $P \neq \overline{P}$, then only one of P, $\overline{P}$ can appear in the sum (as $[P] + [\overline{P}] = 0$)

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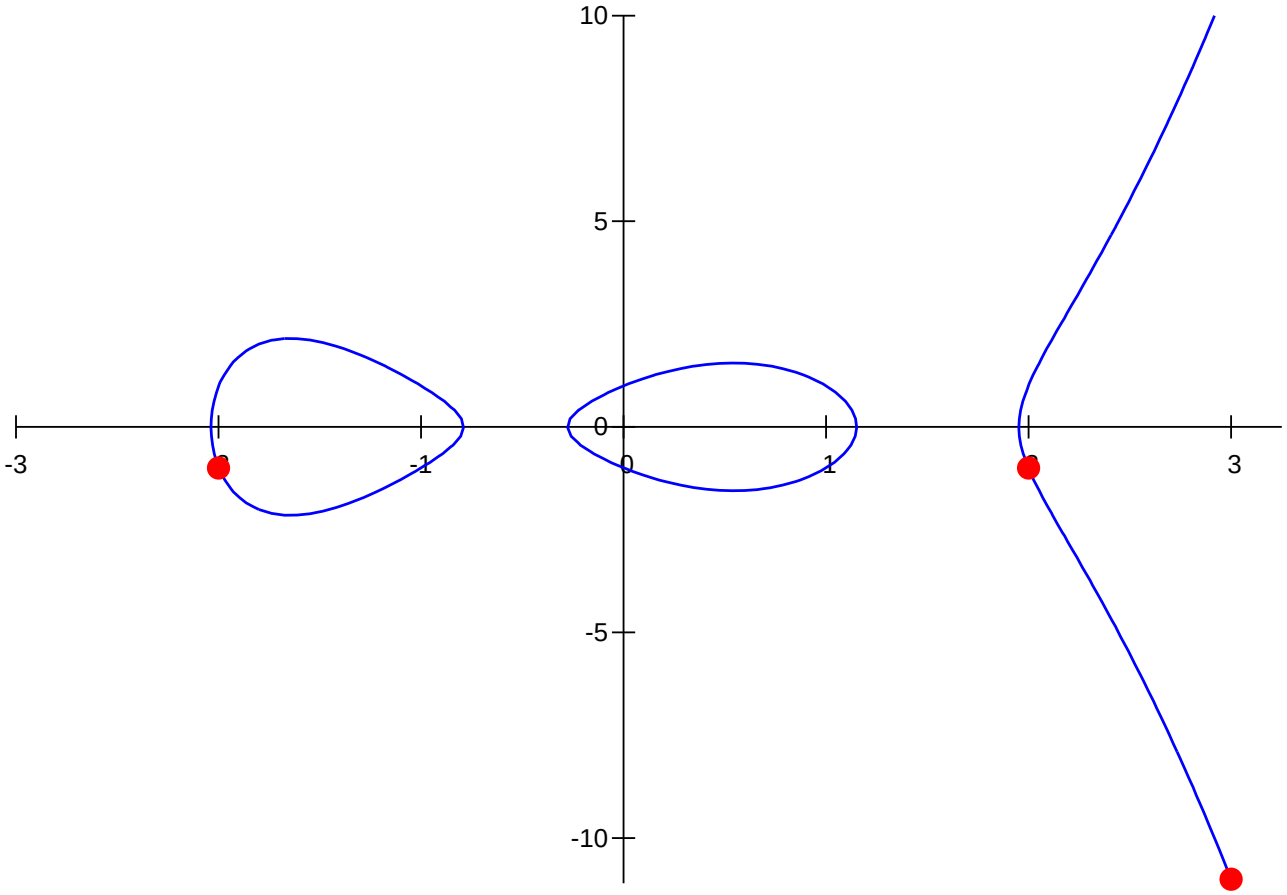
Theorem

Every class in $\mathrm{Jac}_H(\overline{K})$ contains a unique reduced divisor.

Example — Reduction

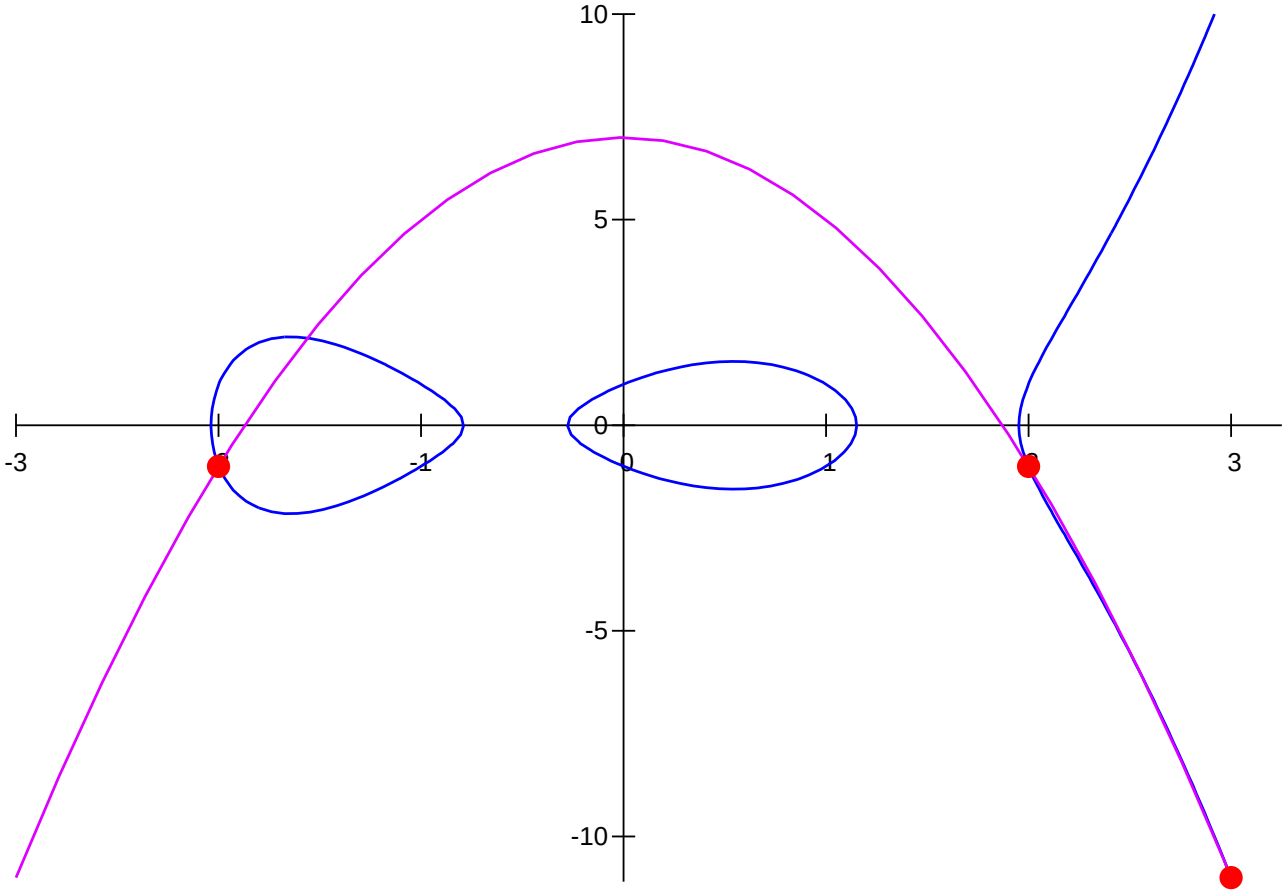
$H : y^2 = x^5 - 5x^3 + 4x - 1$ over $\mathbb{Q}$, $D = [R_1] + [R_2] + [R_3]$ with

$R_1 = (-2, -1)$, $R_2 = (2, -1)$, $R_3 = (3, -11)$



Example — Reduction

R_1, R_2, R_3 all lie on the quadratic $y = -2x^2 + 7$

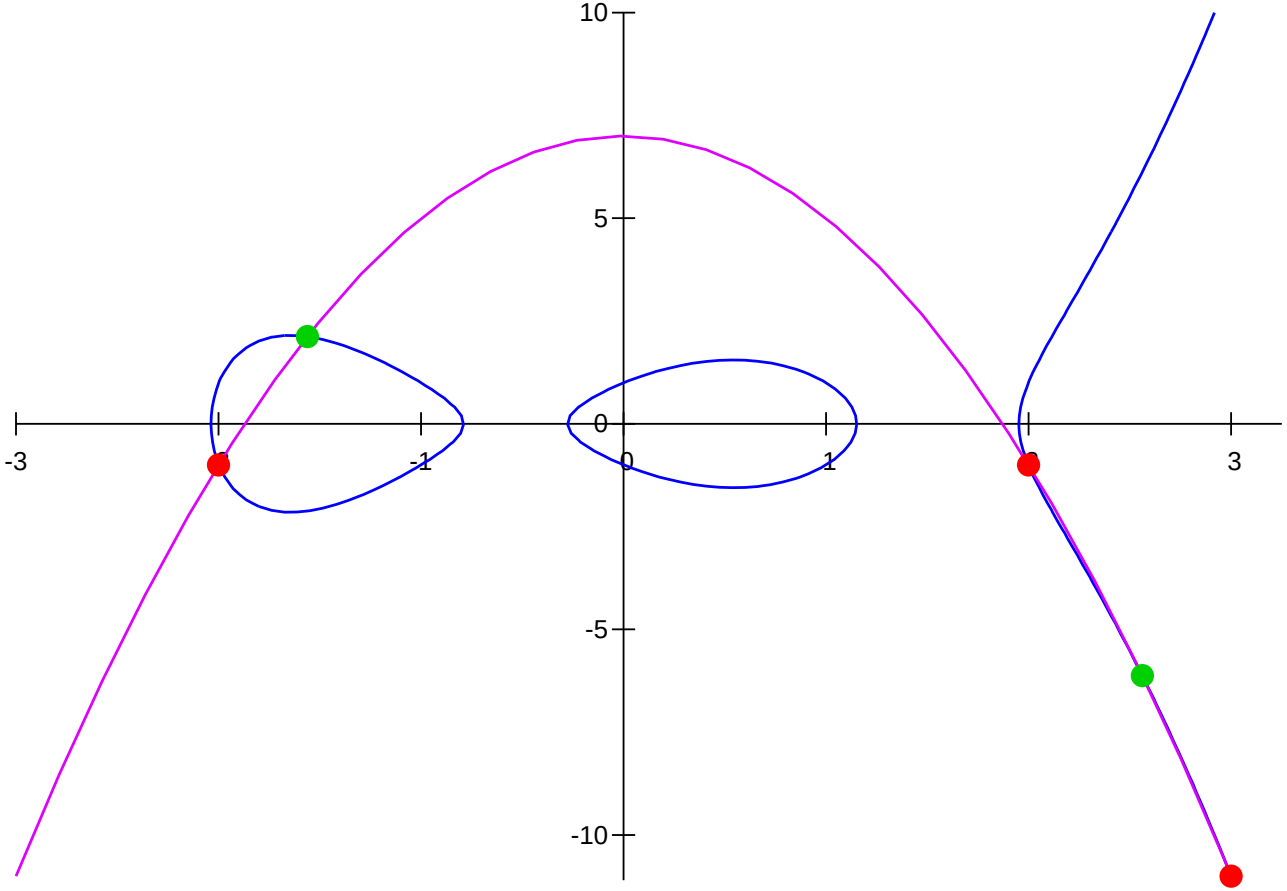


Example — Reduction

This quadratic meets H in the two additional points G_1, G_2 where

$$G_1 = \left(\tfrac{1}{2}(1+\sqrt{17}), -2-\sqrt{17}\right), \quad G_2 = \left(\tfrac{1}{2}(1-\sqrt{17}), -2+\sqrt{17}\right)$$

Thus, $[R_1] + [R_2] + R_3] + [G_1] + [G_2] = 0$ in $\mathrm{Jac}_H(\mathbb{Q})$

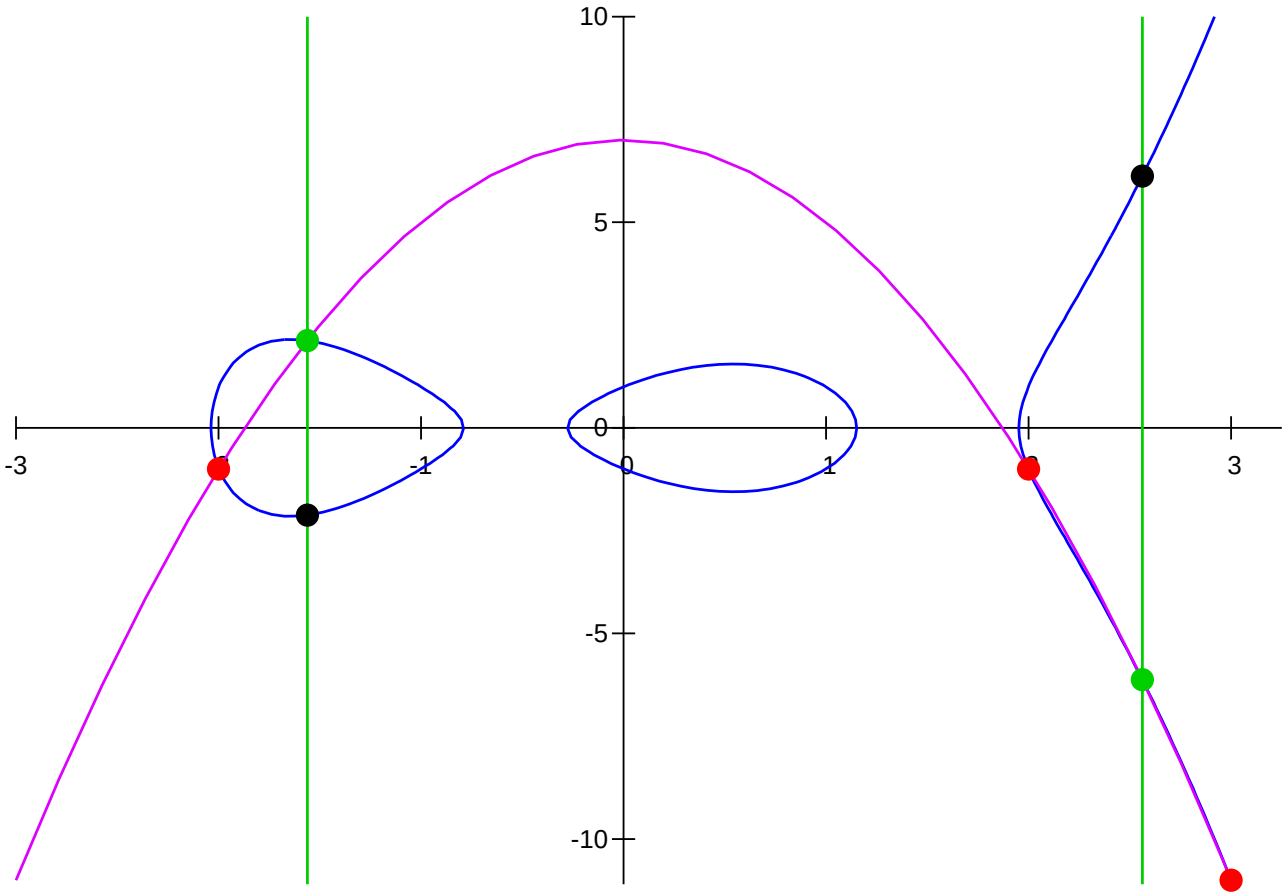


Example — Reduction

$[R_1] + [R_2] + R_3] = [B_1] + [B_2]$ with $B_1 = [\overline{G_1}]$, $B_2 = [\overline{G_2}]$

The reduced divisor in the class of D is $E = [B_1] + B_2]$ where

$B_1 = \left(\frac{1}{2}(1+\sqrt{17}), 2+\sqrt{17}\right)$, $B_2 = \left(\frac{1}{2}(1-\sqrt{17}), 2-\sqrt{17}\right)$,



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After $\left\lceil \frac{r - g}{2} \right\rceil$ steps a reduced divisor is obtained ☺

Mumford Representation

Let $D = \sum_{i=1}^{r} m_i[P_i]$ be a semi-reduced divisor, $P_i = (x_i, y_i)$

The **Mumford representation** of D is $D = (u(x), v(x))$ where

$$u(x), v(x) \in \overline{K}[x], \quad u \text{ monic}, \quad \deg(v) < \deg(u), \quad u \mid v^2 + hv - f$$

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$$u(x) = \prod_{i=1}^{r} (x - x_i)^{m_i}$$

$$\left(\frac{d}{dx}\right)^j \left[v(x)^2 + v(x)h(x) - f(x)\right]_{x=x_i} = 0 \qquad (0 \le j \le m_i - 1)$$

So $u(x_i) = 0$ and $v(x_i) = y_i$ with multiplicity m_i for $1 \le i \le r$

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Example: If $P = (x_0, y_0)$, then $D = [P] = (x - x_0, y_0)$

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while $\deg(u) > g$ *do*

$$u' = \frac{f - vh - v^2}{u}, \quad v' \equiv -v - h \pmod{u'}$$

$$u = u', \quad v = v'$$

end while

return(u', v')

Divisor Reduction — Example

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Recall $D' = \left[\left(\frac{1}{2}(1+\sqrt{17}),\, 2+\sqrt{17}\right)\right] + \left[\left(\frac{1}{2}(1-\sqrt{17}),\, 2-\sqrt{17}\right)\right]$

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$$d = \gcd(u_1, u_2, v_1 + v_2 + h) = s_1 u_1 + s_2 u_2 + s_3(v_1 + v_2 + h)$$
$$u = \frac{u_1 u_2}{d^2}$$
$$v \equiv \frac{1}{d}\big(s_1 u_1 v_2 + s_2 u_2 v_1 + s_3(v_1 v_2 + f)\big) \pmod{u}$$

(In the simplest case above, $d = 1$ and $s_3 = 0$)

Arithmetic in $\mathrm{Jac}_H(\overline{K})$

Input: $D_1 = (u_1, v_1)$, $D_2 = (u_2, v_2)$ reduced

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Methods

- "Vanilla" method just discussed
- Cantor's algorithm ("improved vanilla")
- NUCOMP
- Explicit formulas (if g is small, say $g = 2, 3, 4$)

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Theorem

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Corollary

The group $Jac_H(\mathbb{F}_q)$ of divisor classes defined over $\mathbb{F}_q$ is finite.

Group Structure and Size

$$E(\mathbb{F}_q) \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \quad \text{where} \quad n \mid \gcd(m, q-1)$$

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Hasse: $|E(\mathbb{F}_q)| = q + 1 - t$ with $|t| \leq 2\sqrt{q}$

Hasse-Weil: $|H(\mathbb{F}_q)| = q + 1 - t$ with $|t| \leq 2g\sqrt{q}$

Serre: $|t| \leq g\lfloor 2\sqrt{q} \rfloor$

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For the Jacobian:

$$(\sqrt{q} - 1)^{2g} \leq |\mathrm{Jac}_H(\mathbb{F}_q)| \leq (\sqrt{q} + 1)^{2g}$$

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So $|\mathrm{Jac}_H(\mathbb{F}_q)| \approx q^g$

If we want $q^g \approx 2^{160}$:

g	1	2	3	4
q	2^{160}	2^{80}	$2^{53.33}$	2^{40}

Point Counting Algorithms

Stay for next week's workshop to learn more . . .

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Can also **construct** curves with good group orders via CM method (see Thursday's talks by Drew Sutherland and Bianca Viray)

Discrete Logarithms

Elliptic Curve DLP: given $P, Q \in E(\mathbb{F}_q)$ with $Q = mP$, find m

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For $g = 1, 2$, generic methods are best! (As far as we know . . .)

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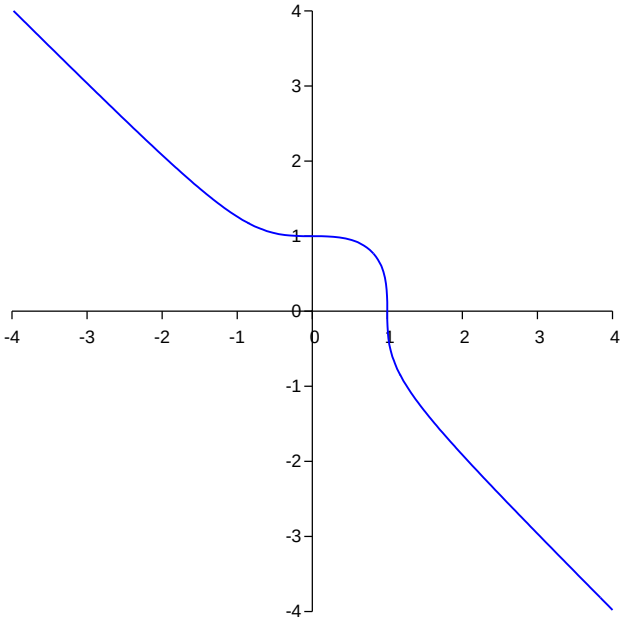
- **Weil Descent**: If $n = kd$ is composite, one may have $\mathrm{Jac}_H(\mathbb{F}_{p^{kd}}) \hookrightarrow \mathrm{Jac}_C(\mathbb{F}_{p^d})$ where C has higher genus — use $n = 1$ or n prime

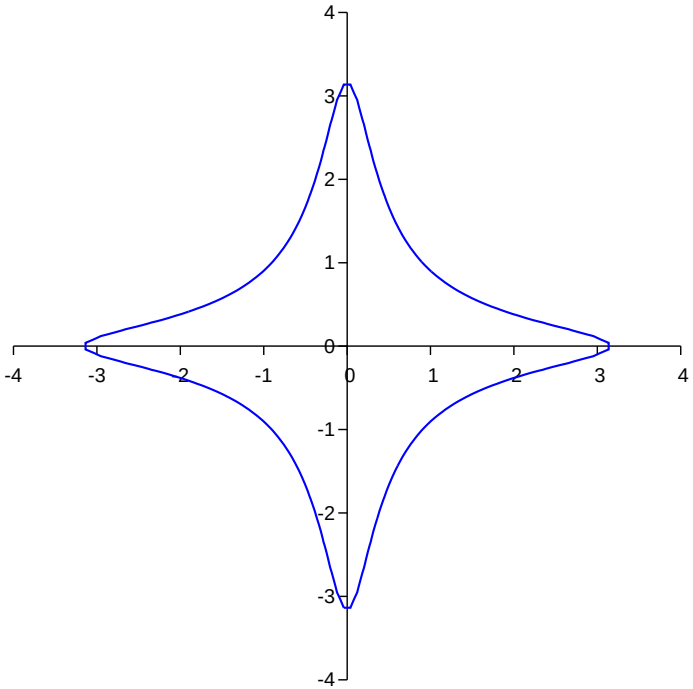
Some Other Models

- **Hessians**: $x^3 + y^3 - 3dxy = 1$
- **Edwards models**: $x^2 + y^2 = c^2(1 + dx^2y^2)$ (q odd) and variations

$x^3 + y^3 = 1$



$x^2+y^2 = 10(1-x^2y^2)$



Even Degree Models

$$H \ : \ y^2 + h(x)y = f(x)$$

- $h(x), f(x) \in K[x]$
- $\deg(h) = g + 1$ if $\mathrm{char}(K) = 2$; $h(x) = 0$ if $\mathrm{char}(K) \neq 2$
- $\deg(f) = 2g + 2$ is even
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Elliptic quartic, $\mathrm{char}(K) \neq 2, 3$:

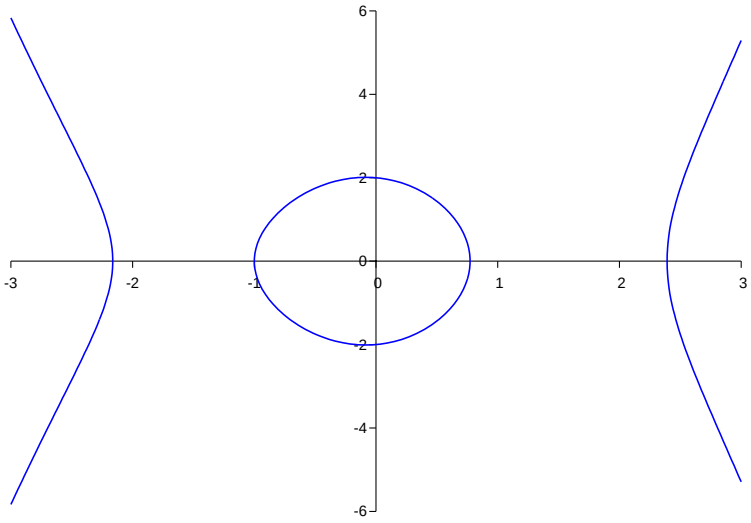
$$y^2 = x^4 + ax^2 + bx + c \quad (a, b, c \in K)$$

($b = 0$: **Jacobi** Quartic)

Examples

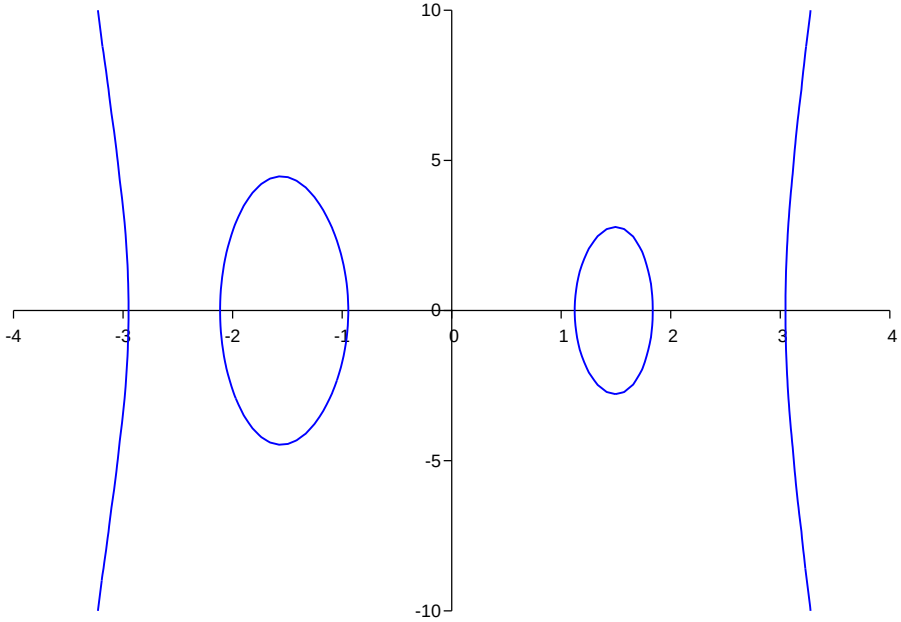
$$E \ : \ y^2 = x^4 - 6x^2 + x + 6$$

$$g = 1$$



$$H \ : \ y^2 = x^6 - 13x^4 + 44x^2 - 4x - 1$$

$$g = 2$$



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$*\ *\ *$ **Thank You!** $*\ *\ *$