Operator Equations in High Dimensions I

Sparse Tensor Methods for Operator Equations with Stochastic Data

Ch. Schwab
Seminar für Angewandte Mathematik
ETH Zürich, Switzerland

Tobias von Petersdorff (UM College Park), R.A. Todor (SAM)

H. Harbrecht (Bonn) and R. Schneider (CAU, Kiel)

Scientific Computing Days Montreal, April 16+17, 2007

IHP Network "Breaking Complexity" EC contract number HPRN-CT-2002-00286
Swiss Federal Office for Science and Education grant No. BBW 02.0418

Numerical models in engineering can be solved with high accuracy

if input data are known exactly.

Often, however,

input data are not known exactly

and

accurate numerical solutions are of limited use.

- Mathematical description of uncertainty in input data and solution?
- How to *propagate* data uncertainty through an engineering FEM simulation?
- How to process statistical information in FEM?

Goal:

given statistics of input data, compute (deterministic) solution statistics.

Tool:

Formulation and solution of *Stochastic Partial Differential Equation (SPDE)*

Basic Problem: Operator Equation w. Stochastic Data

Find $u : \Omega \ni \omega \longrightarrow V$ such that

$$Au = f(\cdot, \omega), \qquad f : \Omega \ni \omega \longrightarrow V'$$

References

Perturbation Methods; "First Order Second Moment" (FOSM)

J. B. Keller (1964)
M. Kleiber, T.D. Hien (1992)
CS and R.A. Todor (2003)
CS and T. von Petersdorff (2006)

Stochastic Galerkin; Wiener Polynomial Chaos (Karhunen-Loève)

R. G. Ghanem, P. D. Spanos (1991)
I. Babuska , R. Tempone and G. Zouraris, SINUM (2003- 2005)
G. E. Karniadakis, D. Xiu SIAM J. Sci. Comp. (2002)
Matthies and Keese, CMAME (2005)
F. Nobile, R. Tempone and F. Webster, SINUM (2006)

Outline

1 Random fields, statistics

2 Stochastic boundary value problem (sBVP)

3 Stochastic Operator Equations

4 Example: Stochastic boundary integral equation (sPDE)

5 Sparse Monte Carlo FEM

6 Sparse Tensor Product FEM

7 Application: Sparse Wavelet BEM for 2nd Moment Analysis in Random Domains

8 Conclusions

Random fields, statistics

$$D \subset \mathbb{R}^d \quad \text{bounded domain,} \quad \Gamma = \partial D = \Gamma_0 \cup \Gamma_1 \quad \text{Lipschitz,}$$
$$(\Omega, \Sigma, P) \quad \text{probability space}$$

Random fields on Γ, D:

X separable Hilbert space. $u(x, \omega)$ *random field* iff

$$u \in L^0(\Omega, X) := \{u(x, \omega) : \quad \Omega \to X| \quad \Omega \ni \omega \to \|u(\cdot, \omega)\|_X \text{ is } P\text{-measurable } \}$$

A random field $u \colon \Omega \to X$ is in $L^1(\Omega, X)$ if $\omega \mapsto \|u(\omega)\|_X$ is integrable so that

$$\|u\|_{L^1(\Omega,X)} := \int_\Omega \|u(\omega)\|_X \, dP(\omega) < \infty$$

In this case the Bochner integral

$$\mathbb{E}u := \int_\Omega u(\omega) dP(\omega) \in X$$

exists and we have

$$\|\mathbb{E}u\|_X \leq \|u\|_{L^1(\Omega,X)} . \tag{1}$$

$B : X \to Y$ continuous, linear.

$u \in L^k(\Omega, X)$ random field in $X \implies v(\omega) = Bu(\omega) \in L^k(\Omega, Y)$

$$\|Bu\|_{L^k(\Omega,Y)} \leq C \|u\|_{L^k(\Omega,X)}$$

and

$$B \int_\Omega u \, dP(\omega) = \int_\Omega Bu \, dP(\omega).$$

Statistical moments of u: for any $k \in \mathbb{N}$ need k-fold tensor product spaces

$$X^{(k)} = \underbrace{X \otimes \cdots \otimes X}_{k\text{-times}},$$

equipped with natural norm $\| \circ \|_{X^{(k)}}$:

$$\forall u_1, \ldots, u_k \in X \qquad \|u_1 \otimes ... \otimes u_k\|_{X^{(k)}} = \|u_1\|_X ... \|u_k\|_X$$

For $u \in L^k(\Omega, X)$ consider random field

$$u^{(k)} = u(\omega) \otimes \cdots \otimes u(\omega) \in L^1(\Omega, X^{(k)})$$

and

$$\begin{aligned} \left\| u^{(k)} \right\|_{L^1(\Omega, X^{(k)})} &= \int_\Omega \|u(\omega) \otimes \cdots \otimes u(\omega)\|_{X^{(k)}} \, dP(\omega) \\ &= \int_\Omega \|u(\omega)\|_X \cdots \|u(\omega)\|_X \, dP(\omega) = \|u\|^k_{L^k(\Omega,X)} \end{aligned} \tag{2}$$

Define k-th moment (k-point correlation function) $\mathcal{M}^k u$ as expectation of $u \otimes \cdots \otimes u$:

Definition 0

For $u \in L^k(\Omega, X)$ for some integer $k \geq 1$, the k-th moment of $u(\omega)$ is defined by

$$\mathcal{M}^k u = \mathbb{E}[\underbrace{u \otimes \ldots \otimes u}_{k-times}] = \int_{\omega \in \Omega} \underbrace{u(\omega) \otimes \ \ldots \otimes u(\omega)}_{k-times} \, dP(\omega) \in X^{(k)} \tag{3}$$

Application: Covariance of $u \in L^2(\Omega, V)$, V separable and reflexive.

$$\mathrm{C}[u] = \mathbb{E}\left[(u - \mathbb{E}u) \otimes (u - \mathbb{E}u)\right] \in V \otimes V$$

If u "sufficiently regular":
Covariance:

$$\mathrm{C}[u](x, x') = \int_\Omega (u(x, \omega) - \mathbb{E}u(x))(u(x', \omega) - \mathbb{E}u(x'))dP(\omega), \qquad x, x' \in D.$$

k-th Moment (k-point correlation function): if $u \in L^k(\Omega, V)$, then

$$\begin{cases} \mathcal{M}^{(k)} u = \mathbb{E}[u \otimes \ldots \otimes u] \in V^{(k)} := V \otimes \ldots \otimes V : \\ \mathcal{M}^{(k)} u(x_1, \ldots, x_k) := \displaystyle\int_\Omega u(x_1, \omega) \otimes \ldots \otimes u(x_k, \omega) \, dP(\omega) \end{cases}$$

Stochastic Operator Equation

Given $A : V \to V'$ linear, bounded, $f \in L^1(\Omega, V')$, find $u \in L^1(\Omega, V)$:

$$Au = f$$

Assume ex. $\alpha > 0$ and $T : V \to V'$ compact such that

$$\forall v \in V : \ \big\langle (A + T)\, v, v \big\rangle \geq \alpha \, \|v\|_V^2 \tag{4}$$

and

$$\ker A = \{0\} \tag{5}$$

Proposition 1

Assume (4) and (5). Then for every $f \in L^0(\Omega, V')$ exists a unique $u \in L^0(\Omega, V)$ solution of $Au = f$.

Statistics

Mean Field: if $u \in L^1(\Omega, V) \begin{cases} E_u \in V : \\ E_u(x) := \int_\Omega u(x,\omega)\, dP(\omega) \end{cases}$

Covariance: if $u \in L^2(\Omega, V) \begin{cases} C[u] \in V \otimes V : \\ C[u](x,y) := \int_\Omega (u(x,\omega) - E_u(x))(u(y,\omega) - E_u(y))\, dP(\omega) \end{cases}$

Variance: $(\mathrm{Var} u)(x) = \mathbb{E}[u^2](x) - (\mathbb{E}[u](x))^2 = (\mathcal{M}^{(2)}[u])(x,x) - (\mathbb{E}[u](x))^2$

kth Moment: if $u \in L^k(\Omega, V) \begin{cases} \mathcal{M}^{(k)} u \in V^{(k)} := V \otimes \ldots \otimes V : \\ \mathcal{M}^{(k)} u(x_1, \ldots, x_k) := \int_\Omega u(x_1,\omega) \otimes \ldots \otimes u(x_k,\omega)\, dP(\omega) \end{cases}$

Proposition 2

Assume (4) and (5). Then for every $f \in L^k(\Omega, V')$ holds $u \in L^k(\Omega, V)$.

Example: Stochastic Dirichlet Problem

$D \subset \mathbb{R}^3$ bounded, Lipschitz.

$$\Delta U = 0 \text{ in } D$$

subject to Dirichlet boundary conditions

$$\gamma_0 U = U|_\Gamma = u \text{ on } \Gamma.$$

Given

$$u \in L^k(\Omega, H^{\frac{1}{2}}(\Gamma)), \qquad k \geq 0,$$

ex. unique solution

$$U(x, \omega) \in L^k(\Omega, H^1(D)) \qquad \text{(Sch. \& Todor 2003).}$$

Example: BEM for Stochastic Dirichlet Problem

$$U(x,\omega) = (SL\sigma)(x,\omega) := \int_\Gamma e(x,y)\,\sigma(y,\omega) ds_y .$$

$$V = H^{-1/2}(\Gamma), \quad \sigma(x,\omega) : \Omega \to H^{-1/2}(\Gamma) \quad \text{random flux}$$

Fubini: SL and $\mathcal{M}^{(1)}$ commute. Hence

$$\mathbb{E}[U] = \mathcal{M}^{(1)}[U] = \mathcal{M}^{(1)}[SL\sigma] = SL\left[\mathcal{M}^{(1)}[\sigma]\right] = SL\,[\mathbb{E}[\sigma]]$$

where the mean field $\mathbb{E}[\sigma] = \mathcal{M}^{(1)}[\sigma] \in H^{-\frac{1}{2}}(\Gamma)$ satisfies first kind deterministic BIE

$$S\mathbb{E}[\sigma] = \mathbb{E}[u] \in H^{\frac{1}{2}}(\Gamma)\,. \tag{6}$$

Unique Solvability (Nédélec and Planchard (1973)): ex. $c_S > 0$ such that

$$\forall \sigma \in H^{-1/2}(\Gamma) : \quad \langle \sigma, S\sigma \rangle \geq c_S \,\|\sigma\|^2_{H^{-1/2}(\Gamma)}$$

Example: BEM for Stochastic Dirichlet Problem

If in the stochastic Dirichlet problem $u \in L^2\big(\Omega, H^{\frac{1}{2}}(\Gamma)\big)$ and $\mathbb{E}[u] = 0$, then $U \in L^2\big(\Omega, H^1(D)\big)$ and

$$C[U] = \mathcal{M}^{(2)}U = \mathcal{M}^{(2)}(SL\sigma) = (SL \otimes SL)\mathcal{M}^{(2)}\sigma = \int\limits_{\Gamma}\int\limits_{\Gamma} e(x,z)\, e(y,w)\, C[\sigma](z,w) ds_z\, ds_w\,,$$

where

$$C[\sigma] \in H^{-\frac{1}{2},-\frac{1}{2}}(\Gamma \times \Gamma) := H^{-\frac{1}{2}}(\Gamma) \otimes H^{-\frac{1}{2}}(\Gamma)$$

satisfies the first kind BIE

$$(S \otimes S)\, C[\sigma] = C[u] \in H^{\frac{1}{2},\frac{1}{2}}(\Gamma \times \Gamma)\,.$$

Solvability:

$$\forall C[\sigma] \in H^{-\frac{1}{2},-\frac{1}{2}}(\Gamma \times \Gamma): \quad \langle (S \otimes S)C[\sigma], C[\sigma]\rangle \geq c_S^2 \|C[\sigma]\|^2_{H^{-\frac{1}{2},-\frac{1}{2}}(\Gamma\times\Gamma)}$$

Goal of Computation

For the operator equation

$$Au = f$$

with $f \in L^k(\Omega, V)$,

$$\text{given } \mathcal{M}_f^{(k)}, \quad \text{find } \mathcal{M}_u^{(k)}.$$

Approaches:

- Monte-Carlo Galerkin FEM ("Collocation in ω"): dense and sparse
- Sparse Wavelet FEM for deterministic approximation of $\mathcal{M}^{(k)}$

Monte Carlo - I

Given data ensemble

$$\{f(\omega_j), \quad j = 1, ..., M\} \subset V'$$

generate (in parallel) solution ensemble

$$\{u(\omega_j), \quad j = 1, ..., M\} \subset V$$

Theorem 3

Assume (4) and (5) and that $f \in L^{2k}(\Omega, V')$.
Estimate $\mathcal{M}^{(k)}u$ by the k-th moment of ensemble $\{u(\omega_j) : j = 1, ..., M\}$, i.e. by

$$\bar{E}^M_{\mathcal{M}^{(k)}u} := \overline{u \otimes \cdots \otimes u}^M = \frac{1}{M} \sum_{j=1}^{M} u(\omega_j) \otimes ... \otimes u(\omega_j) \in V^{(k)}.$$

Then ex. $C(k) > 0$ such that for every $M \geq 1$ and every $0 < \varepsilon < 1$ holds

$$P\left(\|\mathcal{M}^{(k)}u \ - \bar{E}^M_{\mathcal{M}^{(k)}u}\|_{V\otimes...\otimes V} \leq C \frac{\|\mathcal{M}^{2k}(f)\|^{1/2}_{V'^{(2k)}}}{\sqrt{\varepsilon M}} \right) \geq 1 - \varepsilon \tag{7}$$

Monte Carlo - II

Lemma (Law of iterated logarithm in Hilbert spaces):

V separable Hilbert and $X \in L^2(\Omega, V)$. Then

$$\limsup_{M\to\infty} \frac{\left\|\overline{X}^M - E(X)\right\|_V}{(2M^{-1}\log\log M)^{1/2}} \le \|X - E(X)\|_{L^2(\Omega,V)} \quad \text{with probability 1.}$$

Proof: Classical law of iterated logarithm: for real valued $Y(\omega)$ holds

$$\limsup_{M\to\infty} \frac{\left|\overline{Y}^M - E(Y)\right|^2}{2M^{-1}\log\log M} = \mathrm{Var}Y \quad \text{with probability 1.} \tag{8}$$

Let $Z := X - E(X)$. V separable $\Rightarrow$ w.l.o.g $V = \ell^2 = \mathrm{span}\{e_j\}_{j=1}^{\infty}$ and $Y := (e^j, Z) = Z_j \in \mathbb{R}$. Apply (8) with

$$\mathrm{VarY} = (e^j \otimes e^j, \mathcal{M}^2 Z) = (\mathcal{M}^2 Z)_{j,j}.$$

Add estimates for $j = 1, 2, \ldots$ and obtain

$$\limsup_{M\to\infty} \frac{\sum_{j=1}^{\infty} |Z_j|^2}{2M^{-1}\log\log M} \le \sum_{j=1}^{\infty} (\mathcal{M}^2 Z)_{j,j} \quad \text{with probability 1.}$$

Monte Carlo - II

Application: P-a.s. convergence of MCM (*Semidiscrete Case !*)

Theorem 4

Let $f \in L^{2k}(\Omega, V')$. Then

$$\limsup_{M\to\infty} \frac{\left\| \bar{E}^M_{\mathcal{M}^k u} - \mathcal{M}^k u \right\|_{V^{(k)}}}{(2M^{-1}\log\log M)^{1/2}} \le C \, \|f\|^k_{L^{2k}(\Omega,V')} \quad \text{with probability 1.}$$

Monte Carlo - III
MCM - convergence in the absence of 2nd Moments

Theorem 5

Let $k \geq 1$ and assume

$$f \in L^{\alpha k}(\Omega, V') \quad \text{for some} \quad \alpha \in (1, 2].$$

Then ex. C such that for every $M \geq 1$ and every $0 < \varepsilon < 1$

$$P\left(\|\bar{E}^M_{\mathcal{M}^k u} - \mathcal{M}^k u\|_{V^{(k)}} \leq C \frac{\|f\|^k_{L^{\alpha k}(\Omega, V')}}{\varepsilon^{1/\alpha} M^{1-1/\alpha}} \right) \geq 1 - \varepsilon \tag{9}$$

So far: MCM assuming that $Au = f$ solved exactly ("Semidiscrete MCM").

Next: Galerkin FEM in V.

Galerkin FEM

Dense sequence of subspaces:

$$V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_\ell \subset V_{\ell+1} \subset \ldots V$$

Galerkin FEM: given $f \in L^k(\Omega, V')$, find

$$u_L(\omega) \in L^k(\Omega, V^L) \text{ such that } \quad \langle v_L, Au_L(\omega)\rangle = \langle v_L, f(\omega)\rangle \qquad \forall v_L \in V_L$$

Galerkin Projection: $G_L : V \to V_L$ defined by

$$\forall v_L \in V_L : \quad \langle AG_L u, v_L\rangle = \langle f, v_L\rangle$$

is stable: ex. $L_0 > 0$ s.t.

$$\forall L \geq L_0 : \qquad \|G_L u\|_V \leq C\|u\|_V$$

and converges quasioptimally:

$$\forall L \geq L_0 \quad \forall v_L \in V_L : \qquad \|u(\omega) - u_L(\omega)\|_V \leq C\|u(\omega) - v\|_V \quad P - \text{a.e.}\,\omega \in \Omega.$$

Convergence Rates

Smoothness Spaces:

$$\{X_s\}_{s\geq 0}, \quad X_0 = V, \quad X_s \subseteq V, \qquad \{Y_s\}_{s\geq 0}, \quad Y_0 = V', \quad Y_s \subseteq V'$$

Regularity:

$$A^{-1} : Y_s \ni f \to u \in X_s, \quad s \geq 0.$$

Convergence Rate:

$$\|u(\omega) - u_L(\omega)\|_V \leq C\Phi(s, N_\ell)\,\|u\|_{X_s} \quad \text{where} \quad \Phi(s, N_\ell) := \sup_{v\in X_s} \inf_{v_\ell \in V_\ell} \frac{\|v - v_\ell\|_V}{\|v\|_{X_s}} \quad .$$

MC Galerkin: given $\{f(\omega_j) : j = 1, ..., M\}$, compute $\{u_L(\omega_j) : j = 1, ..., M\}$ and

$$\bar{E}^{M,L}_{\mathcal{M}^k u} := \frac{1}{M} \sum_{j=1}^{M} \underbrace{u_L(\omega_j) \otimes ... \otimes u_L(\omega_j)}_{k-\text{times}} \in V_L^{(k)}.$$

Work:

$$O(MN_L^k) \qquad \text{where} \quad N_L = \dim V_L \quad \text{DOFs for “mean field” problem.}$$

Wavelet FEM (Cohen, Dahmen, Kunoth, Schneider, ...)

Wavelet Scale:

$$W_0 := V_0, \qquad V_\ell = V_{\ell-1} \oplus W_\ell,\ \ell = 1, 2, \ldots,$$

Sparse Tensor Product Space (Smol'yak, Teml'yakov, Zenger, Griebel,...):

$$\widehat{V}_L^{(k)} = \sum_{\substack{\vec{\ell} \in \mathbb{N}_0^k \\ |\vec{\ell}| \le L}} W_{\ell_1} \otimes W_{\ell_2} \otimes \cdots \otimes W_{\ell_k}\,.$$

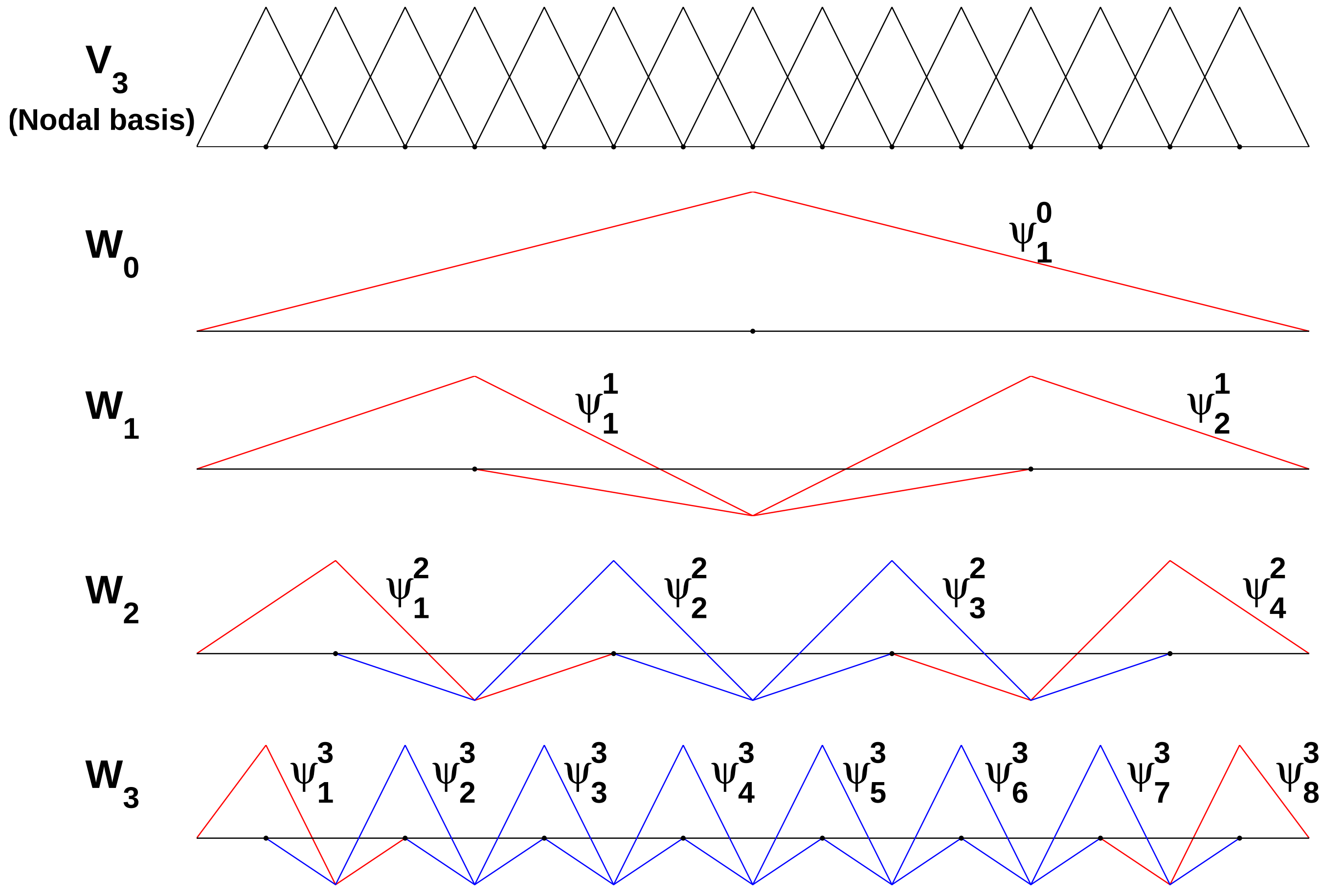
Sparse Projection (quasi-interpolation):

$$\widehat{P}_L^{(k)} : V^{(k)} \to \hat{V}_L^{(k)} \text{ given by } (\widehat{P}_L^{(k)} v)(x) := \sum_{\substack{0 \le \ell_1 + \cdots + \ell_k \le L \\ 1 \le j_\nu \le n_{\ell_\nu}, \nu = 1, \ldots, k}} v_{j_1 \ldots j_k}^{\ell_1 \ldots \ell_j}\, \psi_{j_1}^{\ell_1}(x_1) \ldots \psi_k^{\ell_k}(x_k)$$

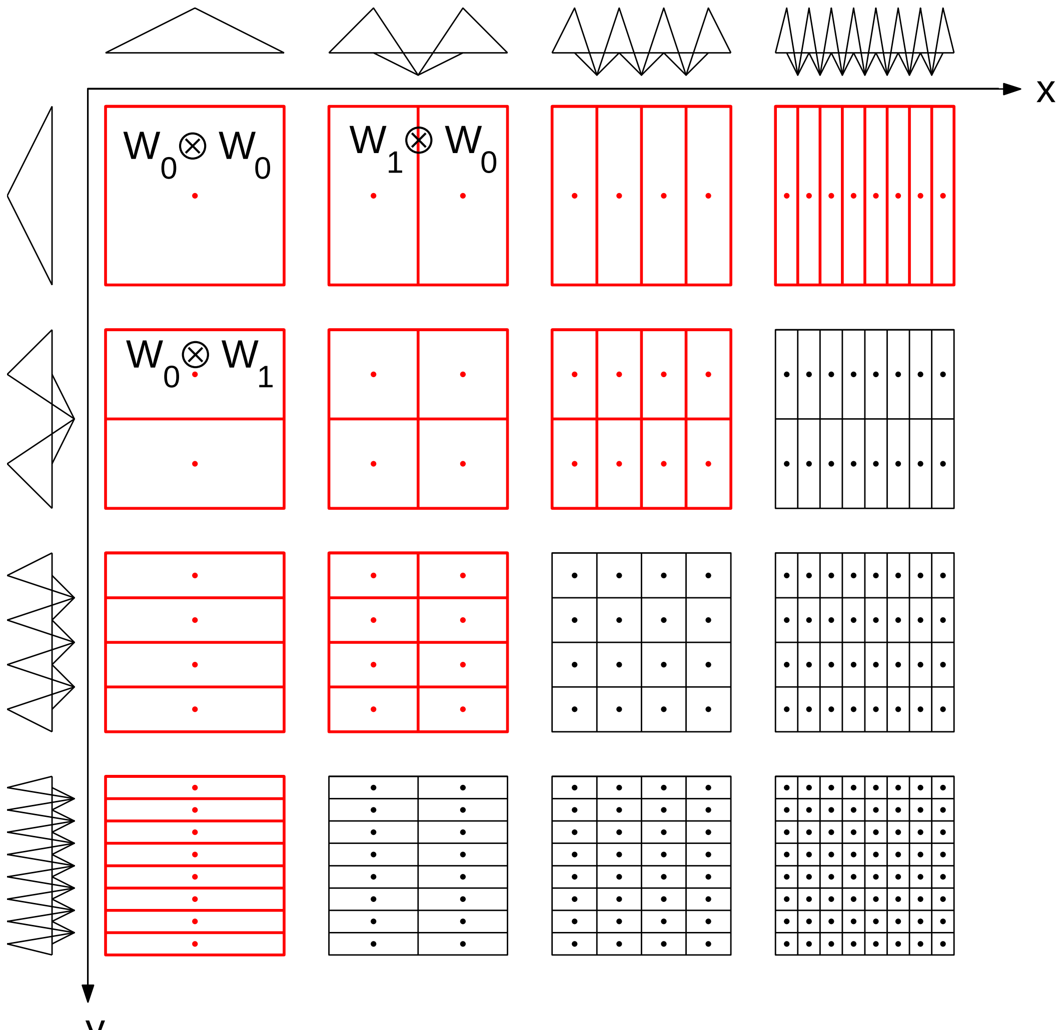
or

$$\widehat{P}_L^{(k)} = \sum_{0 \le \ell_1 + \cdots + \ell_k \le L} Q_{\ell_1} \otimes \cdots \otimes Q_{\ell_k} \quad \text{where} \quad Q_\ell := P_\ell - P_{\ell-1}, \ell = 0, 1, \ldots \text{ and } P_{-1} := 0.$$

Biorthogonal Spline Wavelets in $1-d$, degree $p=1$.



Sparse Tensor Product Space
(Zenger 1990, Griebel & Bungartz Acta Numerica 2004)



Monte Carlo IV – Sparse Monte Carlo FEM

Sparse Tensor Product MC estimate of $\mathcal{M}^k u$:

$$\hat{E}^{M,L}_{\mathcal{M}^k u} := \frac{1}{M} \sum_{j=1}^{M} \widehat{P}^{(k)}_L \left[u_L(\omega_j) \otimes \ldots \otimes u_L(\omega_j) \right] \in V^{(k)}_L .$$

Work:

$$M \ \times \ O(N_L (\log_2 N_L)^{k-1}) \quad \text{operations and} \quad N_L (\log_2 N_L)^{k-1} \quad \text{memory}$$

Theorem 6

Assume $1 < \alpha \le 2$ and

$$f \in L^k(\Omega, Y_s) \cap L^{\alpha k}(\Omega, V') \text{ for some } 0 \le s < s_0 .$$

Then

$$\mathcal{M}^k u \in X_s \otimes \ldots \otimes X_s =: X^{(k)}_s$$

and there is $C(k) > 0$ such that for all $M \ge 1$, $L \ge L_0$ and all $0 < \varepsilon < 1$ holds

$$P\left(\| \hat{E}^{M,L}_{\mathcal{M}^k u} - \mathcal{M}^k u \|_{V^{(k)}} < \lambda \right) \ge 1 - \varepsilon$$

$$\text{with } \lambda = C(k) \left[\Phi(s, N_L) (\log N_L)^{(k-1)/2} \, \|f\|^k_{L^k(\Omega, Y_s)} + \varepsilon^{-1/\alpha} M^{-(1-1/\alpha)} \, \|f\|^k_{L^{\alpha k}(\Omega, V')} \right] .$$

Sparse Tensor Product FEM

Idea:

Compute $\mathcal{M}^k u$ directly, *without* MC

Proposition 7

Assume A satisfies $(4), (5)$ and that $f \in L^k(\Omega, V')$ for $k > 1$.

Then

$$(A \otimes ... \otimes A)Z = \mathcal{M}^k f \,, \tag{10}$$

has a unique solution $Z \in V^{(k)}$ and

$$Z = \mathcal{M}^k u.$$

For $f \in L^k(\Omega, Y_s)$, $s > 0$, holds

$$\|\mathcal{M}^k u\|_{X_s \otimes ... \otimes X_s} \leq C_{k,s} \, \|\mathcal{M}^k f\|_{Y_s \otimes ... \otimes Y_s}, \qquad 0 \leq s < s_0, \quad k \geq 1$$

Regularity of $\mathcal{M}^k u$ in spaces of mixed highest derivative!

Theorem 8

Then for all $L \geq kL_0$ sparse Galerkin approximation $\widehat{Z}_L$ of $\mathcal{M}^k u$ is uniquely defined and

$$\|\mathcal{M}^k u - \widehat{Z}_L\|_{V\otimes\ldots\otimes V} \leq C(k) h_L^s (\log|h_L|)^{(k-1)/2} \|f\|_{L^k(\Omega,Y_s)}, \qquad 0 \leq s < s_0.$$

$\widehat{Z}_L$ can be computed with $O(N_L(\log N_L)^k + 1)$ work and memory.

Note: Tensor product Galerkin FEM gives

$$\|\mathcal{M}^k u - Z_L\|_{V\otimes\ldots\otimes V} \leq C(k) h_L^{s/k} \|f\|_{L^k(\Omega,Y_s)}, \qquad 0 \leq s < s_0$$

in $O(N_L^k)$ memory and work.

Application: Random Solutions on Domains with Stochastic Boundaries

Dirichlet Problem:

$$-\Delta u = f \quad \text{in} \quad D, \qquad u = g \quad \text{on} \quad \partial D.$$

How does u depend on D?

Boundary Perturbation of amplitude $\varepsilon > 0$ in direction $\mathbf{U}$:

$$\mathbf{U}(\mathbf{x}) : \partial D \to \mathbb{R}^3, \qquad ||\mathbf{U}||_2 = 1, \quad \mathbf{U}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) > 0$$

$$\partial D_\varepsilon := \{\mathbf{x} + \varepsilon \mathbf{U}(\mathbf{x}) : \mathbf{x} \in \partial D\}, \qquad D_\varepsilon := \text{interior} \partial D_\varepsilon$$

Specifically:

$$\mathbf{U}(\mathbf{x}) = \kappa(\mathbf{x}) \mathbf{n}(\mathbf{x})$$

with $\kappa(\mathbf{x}) \in C^4(\partial D, \mathbb{R})$

Dirichlet Problem on perturbed domain:

$$-\Delta u_\varepsilon = f \quad \text{in} \quad D_\varepsilon, \qquad u_\varepsilon = g \quad \text{on} \quad \partial D_\varepsilon.$$

Idea: for $\varepsilon > 0$ sufficiently small

$$u_\varepsilon = \bar{u} + \varepsilon du[\mathbf{U}] + \frac{\varepsilon^2}{2} d^2 u[\mathbf{U}, \mathbf{U}] + O(\varepsilon^3)$$

Application: Random Solutions on Domains with Stochastic Boundaries

Thm (Hadamard 1909, F. Murat & J. Simon (1976), J. Sokolowski & P. Zolesio,J. Simon (1980)):

u depends Fréchet-differentiably on D.

The first derivative of u w.r. to D, the *local shape derivative* $du[\mathbf{U}]$, is solution of the Dirichlet problem

$$\Delta du = 0 \text{ in } D, \qquad du = \langle \nabla(g - \bar{u}), \mathbf{U} \rangle = \langle \mathbf{U}, \mathbf{n} \rangle \frac{\partial(g - \bar{u})}{\partial \mathbf{n}} \text{ on } \partial D$$

where $\bar{u}$ is the solution of the Dirichlet Problem on D.

Shape Hessian: bilinear form on pairs of boundary perturbation fields $(\mathbf{U}, \mathbf{U}')$, denoted by

$$d^2u = d^2u[\mathbf{U}, \mathbf{U}'].$$

It is obtained from the Dirichlet problem (Hettlich & Rundell SINUM 2000, Eppler 2003):

$$\Delta d^2u = 0 \text{ in } D,$$

$$d^2u = \langle \mathbf{H}[g - \bar{u}])\mathbf{U}', \mathbf{U} \rangle - \langle \nabla du[\mathbf{U}], \mathbf{U}' \rangle - \langle \nabla du[\mathbf{U}'], \mathbf{U} \rangle \text{ on } \partial D.$$

Application: Random Solutions on Domains with Stochastic Boundaries

Random domain variation:

$$\mathbf{U}(\mathbf{x},\omega) = \kappa(\mathbf{x},\omega)\mathbf{n}(\mathbf{x}),$$

where κ is P-measurable and

$$\kappa(\mathbf{x},\omega) : \Omega \to X = C^k(\partial D, \mathbb{R}), k = 4.$$

Finite second moments of $\kappa(\mathbf{x},\omega)$ in X with respect to P:

$$\mathbb{E}_\kappa(\mathbf{x}) := \int_\Omega \kappa(\mathbf{x},\omega)dP(\omega) = \mathbb{E}\big(\kappa(\mathbf{x},\omega)\big) = 0, \quad \mathbf{x} \in \partial D,$$

and

$$\mathrm{Cor}_\kappa(\mathbf{x},\mathbf{y}) := \int_\Omega \kappa(\mathbf{x},\omega)\kappa(\mathbf{y},\omega)dP(\omega) = \mathbb{E}\big(\kappa(\mathbf{x},\omega)\kappa(\mathbf{y},\omega)\big), \quad \mathbf{x},\mathbf{y} \in \partial D,$$

of $\kappa(\mathbf{x},\omega)$ exist and are known:

$$\mathbb{E}_\kappa = 0 \quad \Longrightarrow \quad \mathrm{Covar}_\kappa = \mathrm{Cor}_\kappa.$$

Application: Random Solutions on Domains with Stochastic Boundaries

Lemma 9

For sufficiently small $\varepsilon > 0$, $u_\varepsilon(\omega)$

$$u_\varepsilon(\mathbf{z},\omega) = \bar{u}(\mathbf{z}) + \varepsilon du(\mathbf{z},\omega) + \frac{\varepsilon^2}{2} d^2u(\mathbf{z},\omega) + \mathcal{O}(\varepsilon^3) \quad \text{for } P - a.e.\ \omega \in \Omega,$$

where $\bar{u} \in H^1(D)$ solves the deterministic Dirichlet problem

$$-\Delta \bar{u} = f \text{ in } D, \qquad \bar{u} = g \text{ on } \partial D,$$

where

$$du(\mathbf{z},\omega) := du[\kappa(\cdot,\omega)\mathbf{n}](\mathbf{z}),$$

and

$$d^2u(\mathbf{z},\omega) := d^2u[\kappa(\mathbf{z},\omega)\mathbf{n}(\mathbf{z}), \kappa(\mathbf{z}',\omega)\mathbf{n}(\mathbf{z}')]|_{\mathbf{z}'=\mathbf{z}}.$$

The remainder term is $\mathcal{O}(\varepsilon^3)$ for P-a.e. $\omega \in \Omega$.

Application: Random Solutions on Domains with Stochastic Boundaries

How to get second moments of $u(\mathbf{x}, \omega)$?

Lemma 10

It holds

$$\mathbb{E}\big(du(\mathbf{z}, \omega)\big) = 0.$$

and, for $\varepsilon > 0$ sufficiently small,

$$\mathbb{E}_u(\mathbf{z}) = \bar{u}(\mathbf{z}) + \mathcal{O}(\varepsilon^2), \qquad \mathbf{z} \in D_\varepsilon.$$

and $\mathrm{Var}_u(\mathbf{z})$ satisfies

$$\mathrm{Var}_u(\mathbf{z}) = \varepsilon^2 \,\mathrm{Var}\big(du(\mathbf{z}, \omega)\big) + \mathcal{O}(\varepsilon^3) = \varepsilon^2 \mathbb{E}\,\left(du(\mathbf{z}, \omega)^2\right) + \mathcal{O}(\varepsilon^3).$$

How to compute $\mathbb{E}\big(du(\mathbf{z}, \omega)^2\big)$ deterministically?

Since $\mathbb{E}(du(\mathbf{x}, \omega)) = 0$,

$$\mathrm{Var}\big(du(\mathbf{z}, \omega)\big) = \mathrm{Cor}\big(du(\mathbf{z}, \omega), du(\mathbf{z}', \omega)\big)\big|_{\mathbf{z}=\mathbf{z}'}.$$

Approximate $\mathrm{Var}\big(du(\mathbf{z}, \omega)\big)$ by trace of two-point correlation of shape gradient du in the "random" direction $\kappa(\mathbf{x}, \omega)\mathbf{n}(\mathbf{x})$.

Application: Random Solutions on Domains with Stochastic Boundaries

Theorem 11

$$\mathrm{Cor}_{du}(\mathbf{z},\mathbf{z}') := \mathrm{Cor}\big(du(\mathbf{z},\omega), du(\mathbf{z}',\omega)\big)$$

is the unique solution in $H^{1,1}(D \times D)$ of the tensor product boundary value problem on $D \times D \subset \mathbb{R}^{2n}$

$$(\Delta_{\mathbf{z}} \otimes \Delta_{\mathbf{z}'})\,\mathrm{Cor}_{du}(\mathbf{z},\mathbf{z}') = 0, \qquad \mathbf{z},\mathbf{z}' \in D,$$

$$\mathrm{Cor}_{du}(\mathbf{x},\mathbf{y}) = \mathrm{Cor}_{\kappa}(\mathbf{x},\mathbf{y})\left[\frac{\partial(g-\bar{u})}{\partial \mathbf{n}}(\mathbf{x}) \otimes \frac{\partial(g-\bar{u})}{\partial \mathbf{n}}(\mathbf{y})\right], \qquad \mathbf{x},\mathbf{y} \in \partial D.$$

Moreover, $\mathrm{Cor}_{du} \in H^{s+1/2,s+1/2}(D \times D)$ provided that $\partial(g-\bar{u})/\partial\mathbf{n} \in H^s(\partial D)$ for some $s \geq 1/2$.

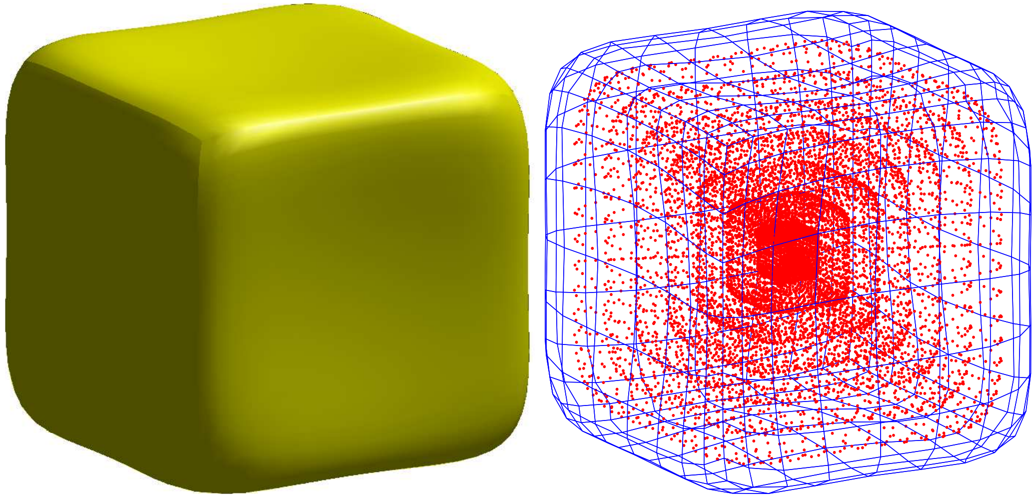


Figure 1: The domain D and the potential evaluation points.

Application: Random Solutions on Domains with Stochastic Boundaries

Numerical Results

$$f = 1, \qquad g = -x^2/2, \qquad u = -x^2/2, \qquad \mathrm{Cor}_\kappa = xyz\exp(-x^2 - y^2 - z^2)$$

J	N_J	$\|\rho - \rho_J\|_{L^2(\partial D)}$	$\|\mathbf{u} - \mathbf{u}_J\|_\infty$	cpu-time
1	24	2.9e-1	5.6e-1	1
2	96	3.5e-1 (0.8)	5.1e-2 (11)	1
3	384	1.7e-1 (2.1)	2.0e-2 (2.5)	2
4	1536	8.4e-2 (2.0)	3.4e-3 (5.9)	9
5	6144	4.2e-2 (2.0)	4.4e-4 (7.9)	47
6	24576	2.1e-2 (2.0)	9.1e-5 (4.8)	413
7	98304	1.0e-2 (2.0)	1.6e-5 (5.6)	2002
8	393216	1.3e-2 (2.0)	3.7e-6 (4.3)	13097

Table 1: Numerical results with respect to the mean field equation.

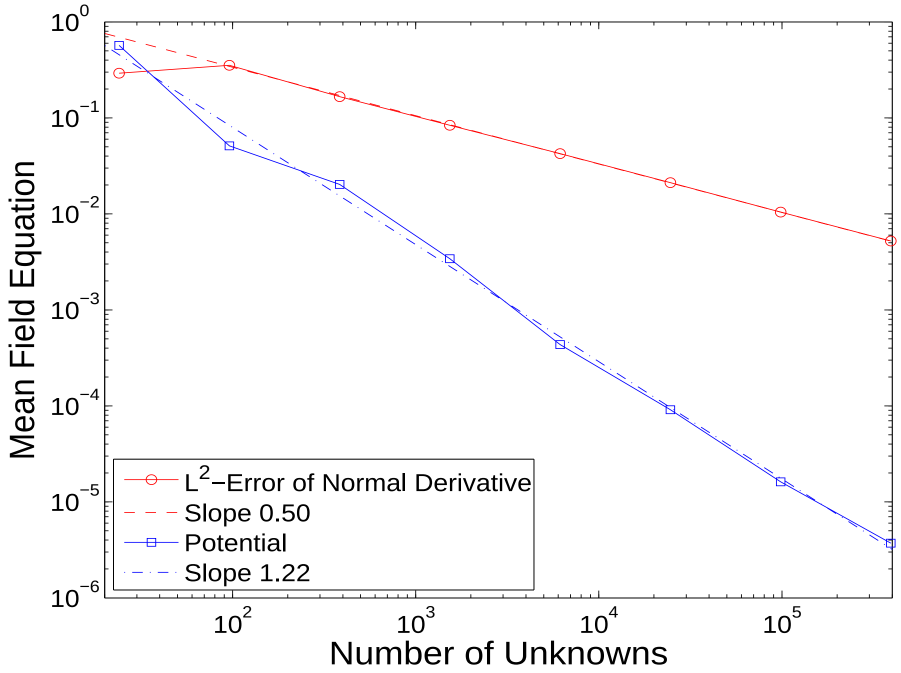
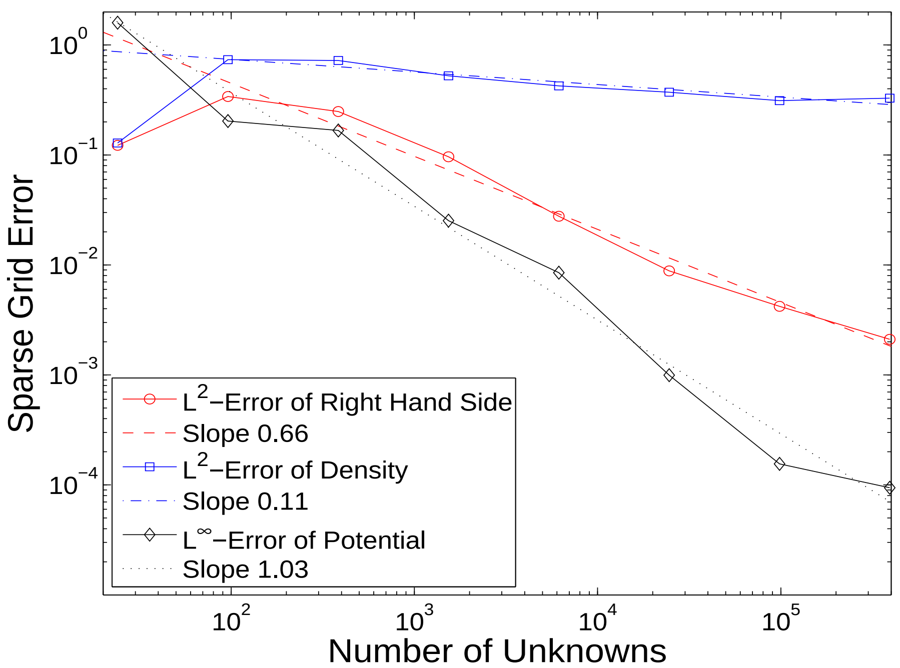


Figure 2: Asymptotic behaviour of the errors for the mean field equation.

Application: Random Solutions on Domains with Stochastic Boundaries

J	$\widehat{N}_J$	$\|Q_J - \widehat{Q}_J\|_{L^2(\partial D\times\partial D)}$	$\|\Sigma_J - \widehat{\Sigma}_J\|_{L^2(\partial D\times\partial D)}$	$\|\mathbf{C}_J - \widehat{\mathbf{C}}_J\|_\infty$	cpu-time
1	252	1.2e-1	1.3e-1	1.6	1
2	1440	3.4e-1 (0.4)	7.3e-1 (0.2)	2.0e-1 (7.8)	1
3	7488	2.5e-1 (1.4)	7.2e-1 (1.0)	1.7e-1 (1.2)	3
4	36864	9.6e-2 (2.6)	5.2e-1 (1.4)	2.5e-2 (6.6)	14
5	175104	2.8e-2 (3.5)	4.2e-1 (1.2)	8.5e-3 (3.0)	124
6	811008	8.8e-3 (3.1)	3.7e-1 (1.1)	1.0e-3 (8.6)	1210
7	3.7 mio	4.2e-3 (2.1)	3.1e-1 (1.2)	1.6e-4 (6.4)	3 hrs
8	16.5 mio	2.1e-3 (2.0)	3.3e-1 (0.9)	9.4e-5 (1.6)	24 hrs

Table 2: Errors in the covariance approximation by the sparse tensor product approach.



Conclusions

- Monte-Carlo Galerkin FEM: framework, convergence analysis
- Sparse Galerkin FEM: regularity in anisotropic spaces; sparse tensor product spaces
- Given data statistics, get solution statistics by deterministic computation
- trade stochasticity and MC for high-dimensionality $+$ deterministic FEM
- Use sparse tensor products of wavelet spaces to avoid $O(N_L^k)$ complexity
- Fast Matrix Vector Multiplication (Sch. & Todor: Numer. Math. 2003)
- a-priori and a-posteriori error estimates, adaptivity
  $\rightarrow$ framework of Cohen, Dahmen, DeVore in tensor product Besov spaces (Nitsche 2006, Stevenson and Sc. 2007)
  $$\mathcal{M}^k(u) \in B_q^\alpha(L_q(D)) \otimes_q \ldots \otimes_q B_q^\alpha(L_q(D))$$
  for arbitrarily large α with
  $$q = [\alpha/2 + 1/2]^{-1} < 1 \quad \text{indep. of} \quad k.$$
- (Fréchet-differentiable) nonlinear problems:
  linearize around "nominal" solution and get 2nd order statistics of the random solution from gradient and Hessian at the "nominal" solution.
- Wavelets **really** needed? No, **for PDEs** "frames" are sufficient...