

Periodic-orbit theory of universal spectral statistics

Alexander Altland, Petr Braun, Fritz Haake, Stefan Heusler

Jon Keating



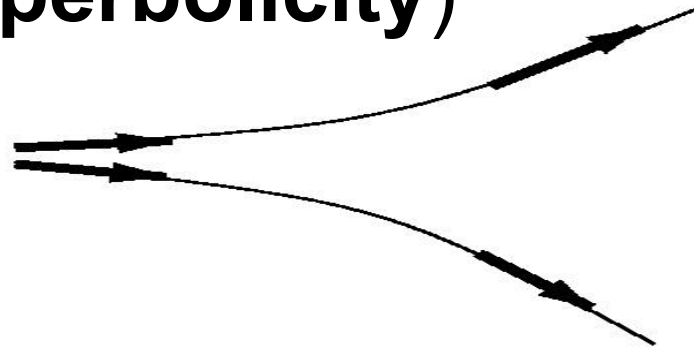
Sebastian Müller

S.H., S.M., A.A., P.B. & F.H, PRL 98, 044103, 2007
J. K. & S. M. , Proc. R. Soc. 2007

Classical chaos

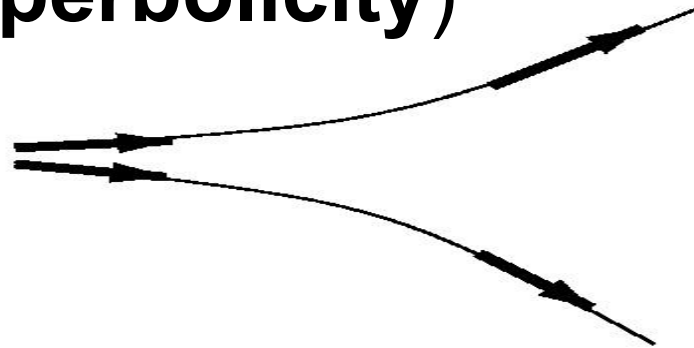
Classical chaos

- sensitive dependence on initial conditions (**hyperbolicity**)

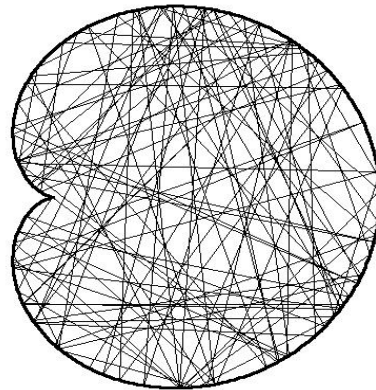


Classical chaos

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- **ergodicity**



Classical chaos

Classical chaos

- **periodic orbits** come in **bunches**

Classical chaos

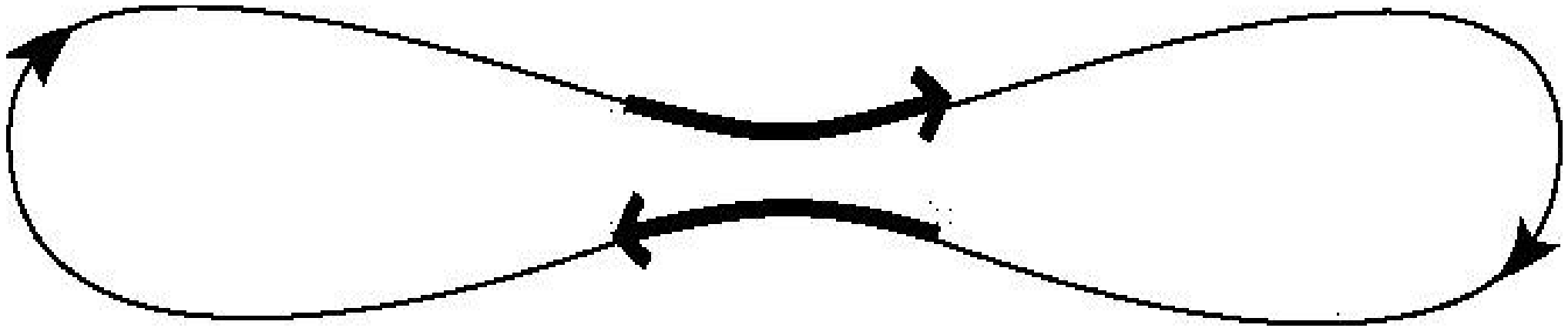
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Sieber/Richter 2001

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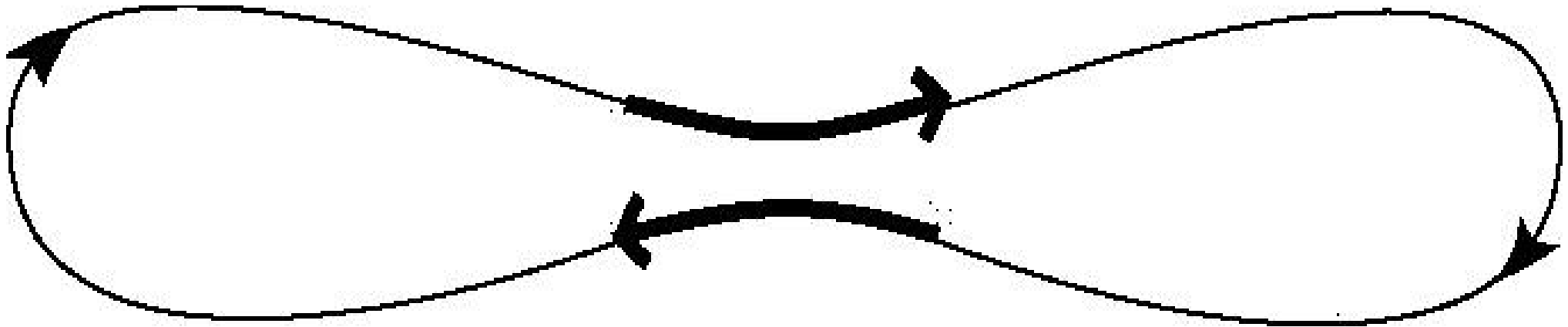
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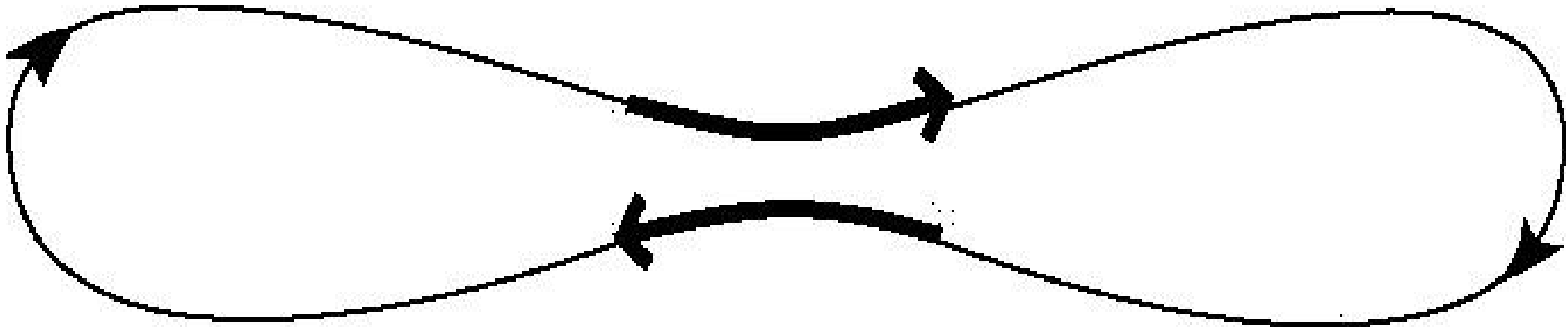


encounter:

Classical chaos

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Sieber/Richter 2001



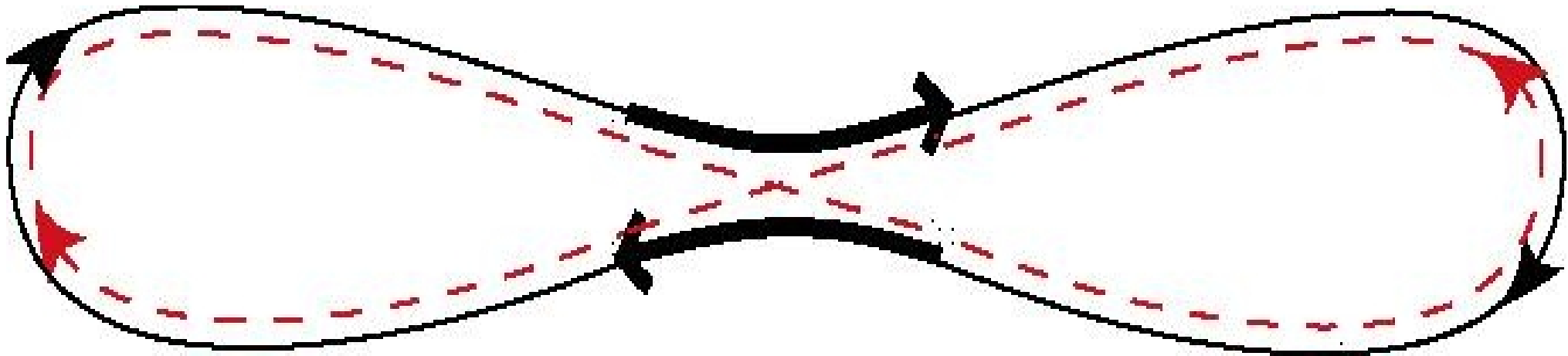
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stretches **close** up to time reversal

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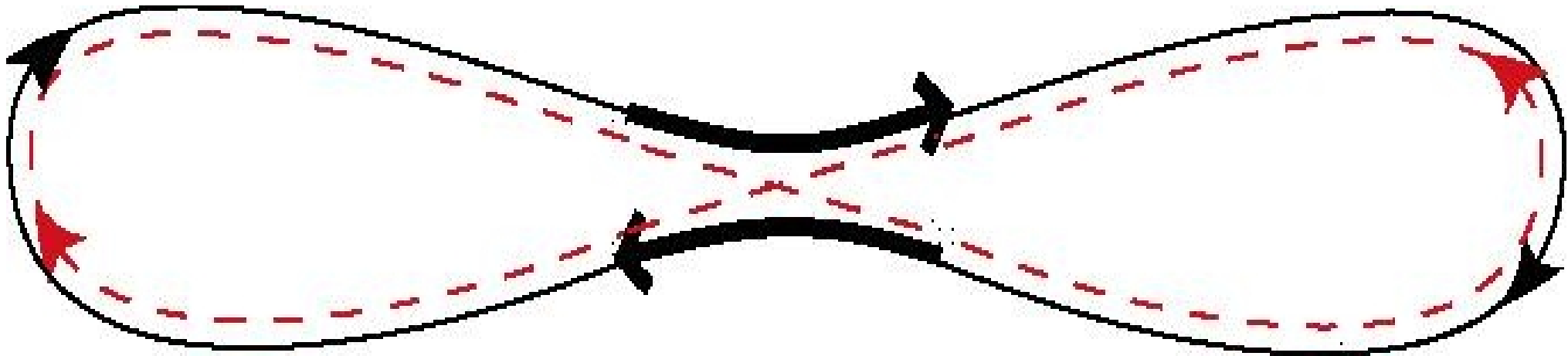
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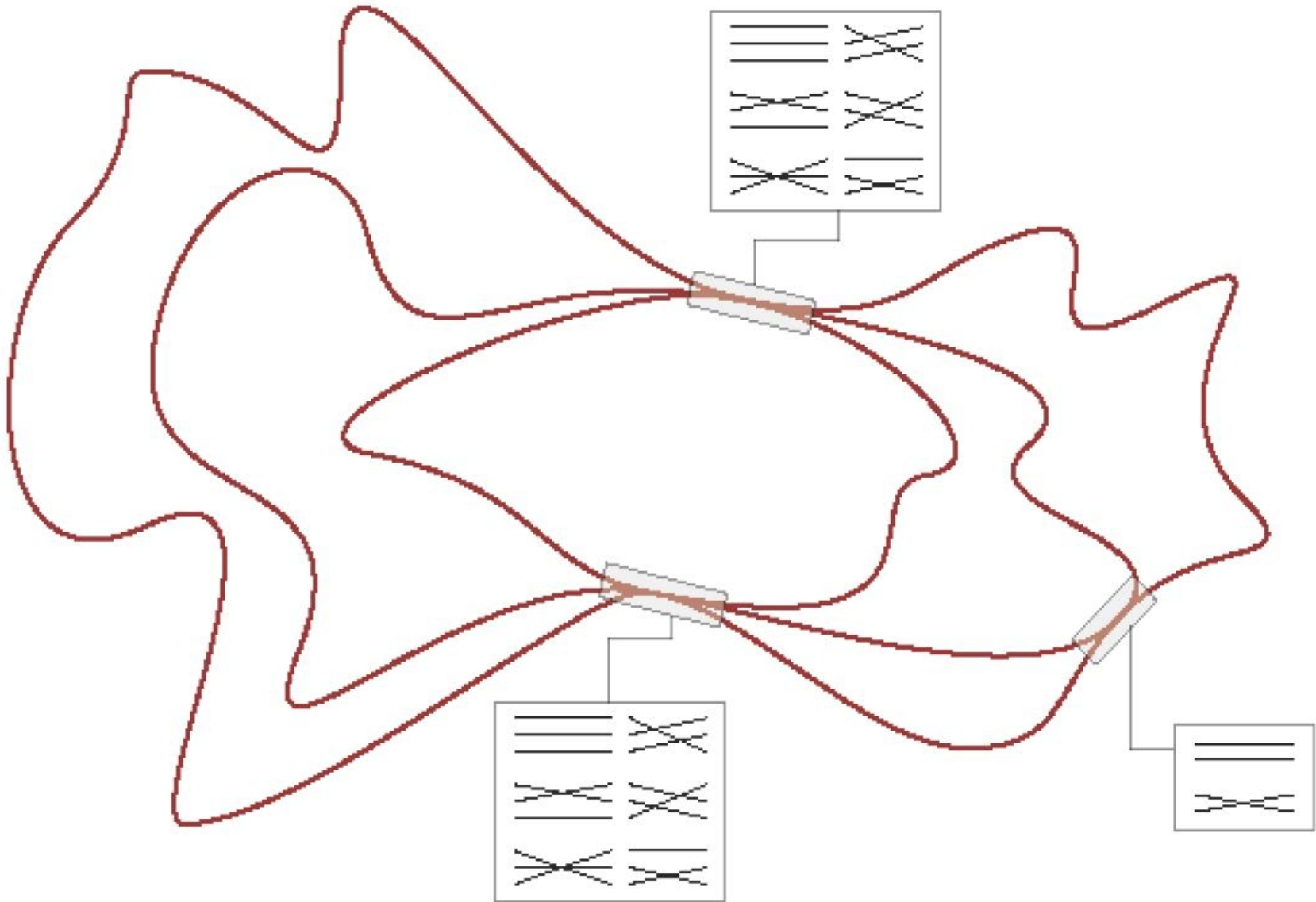
with similar action

Classical chaos

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Quantum chaos

Quantum chaos

Chaotic systems show universal
spectral statistics

Bohigas, Giannoni, Schmit(1984);
Casati et al. (1980); Berry (1987)

Quantum chaos

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e.g.: **correlation function** of level density

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$$R(\epsilon) = \langle \rho(E + \epsilon/2) \rho(E - \epsilon/2) \rangle - 1$$

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$$\begin{aligned} R(\epsilon) &= \langle \rho(E + \epsilon/2) \rho(E - \epsilon/2) \rangle - 1 \\ &= \Re \sum_{n=2}^{\infty} \left(a_n + b_n e^{2\pi i \epsilon} \right) \left(\frac{1}{\epsilon} \right)^n \end{aligned}$$

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(generating function, Riemann/Siegel lookalike)

Generating function

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correlation function obtained from **generating function**

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$$Z = \left\langle \frac{\Delta(E + \gamma) \Delta(E - \delta)}{\Delta(E + \alpha) \Delta(E - \beta)} \right\rangle \quad \Delta(E) = \det(E - H)$$

Generating function

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as

$$R(\epsilon) \propto \Re \frac{\partial^2 Z}{\partial \alpha \partial \beta} \Big|_{\alpha = \beta = \gamma = \delta = \epsilon/2}$$

Semiclassics

$$\text{tr}(E^+ - H)^{-1} \sim -i\pi \bar{\rho}$$

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(Weyl)

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$$\text{tr}(E^+ - H)^{-1} \sim \begin{array}{l} -i\pi\bar{\rho} \\ \text{(Weyl)} \end{array} + \begin{array}{l} \text{sum over } \mathbf{classical\ periodic\ orbits} \\ \text{(Gutzwiller)} \end{array}$$

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action ↓
 sum over **sets of classical periodic orbits (pseudo-orbits) A**

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$\Delta(E)$ should be **real** for real E !

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$$\left(\sum \text{ over pseudo-orbits } > T_H/2 \right) = \left(\sum \text{ over pseudo-orbits } < T_H/2 \right)^*$$

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Riemann-Siegel lookalike

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need **small action differences**

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➔ bunches!

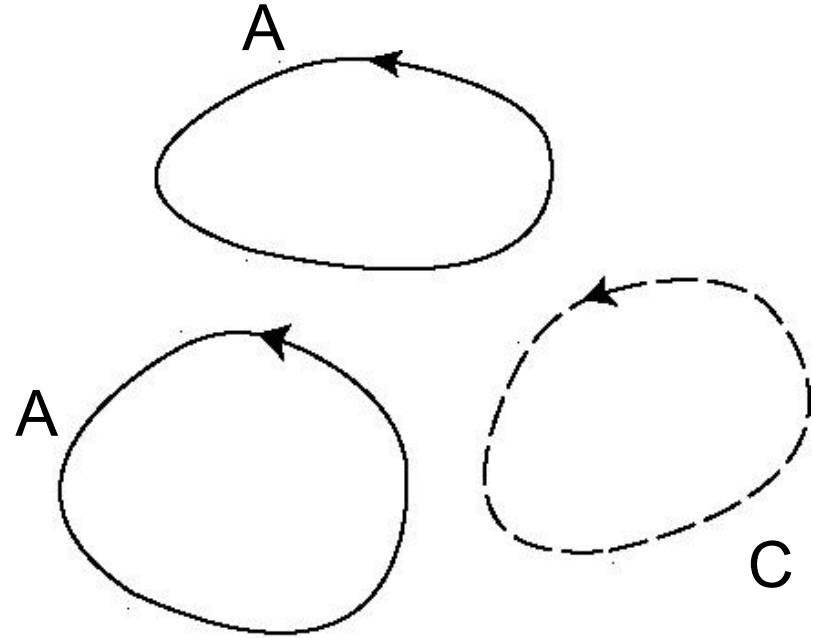


Diagonal contributions

(A,C) contain
same orbits as (B,D):

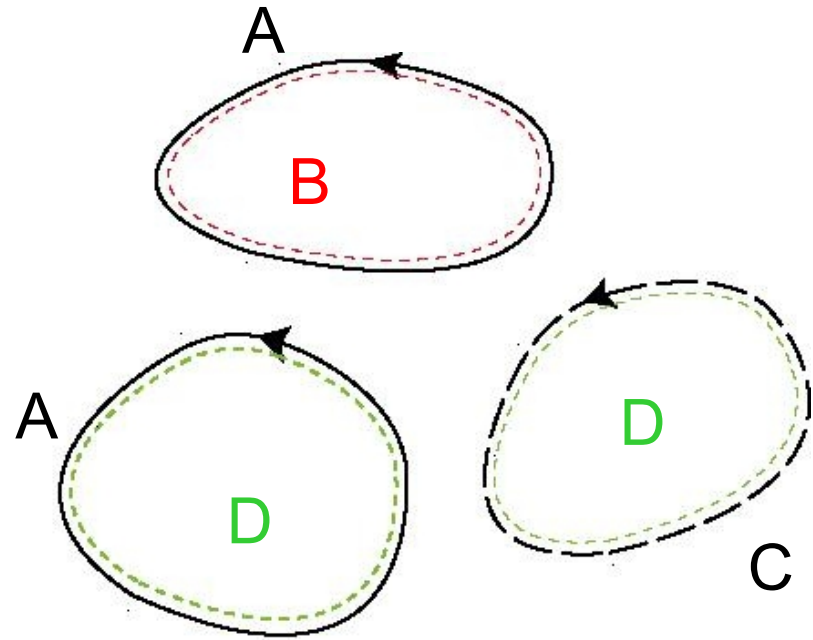
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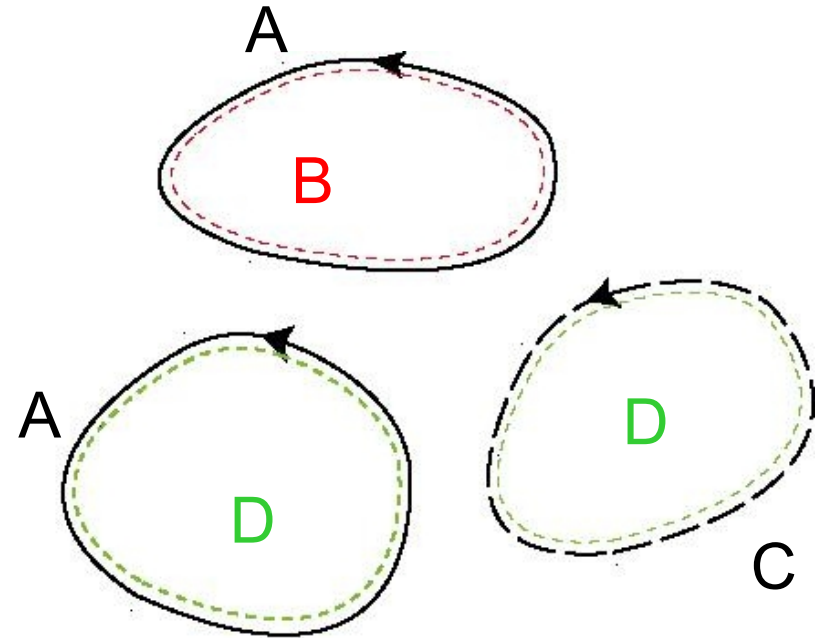
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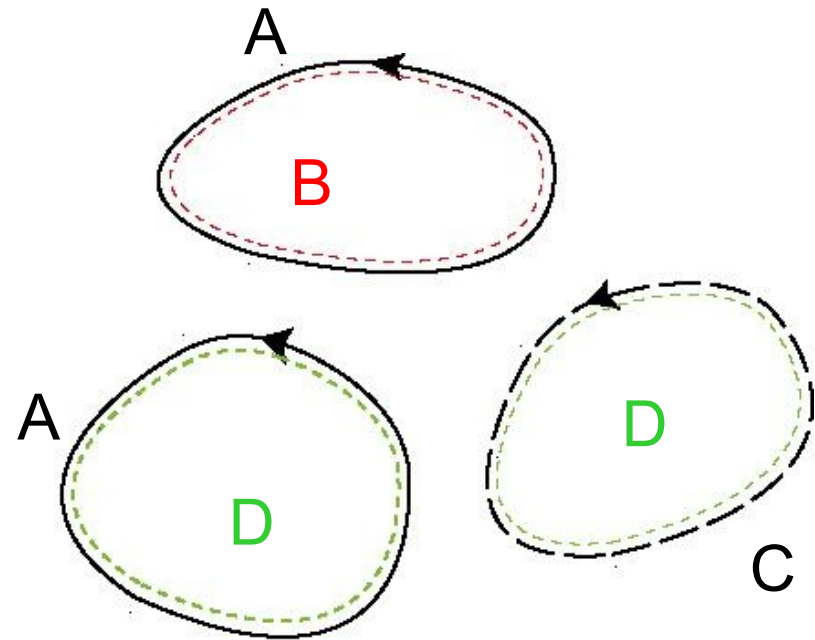
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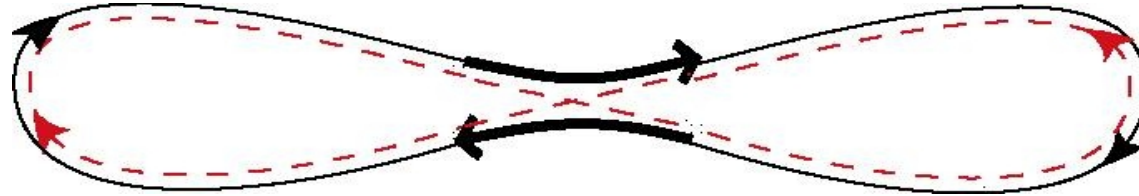


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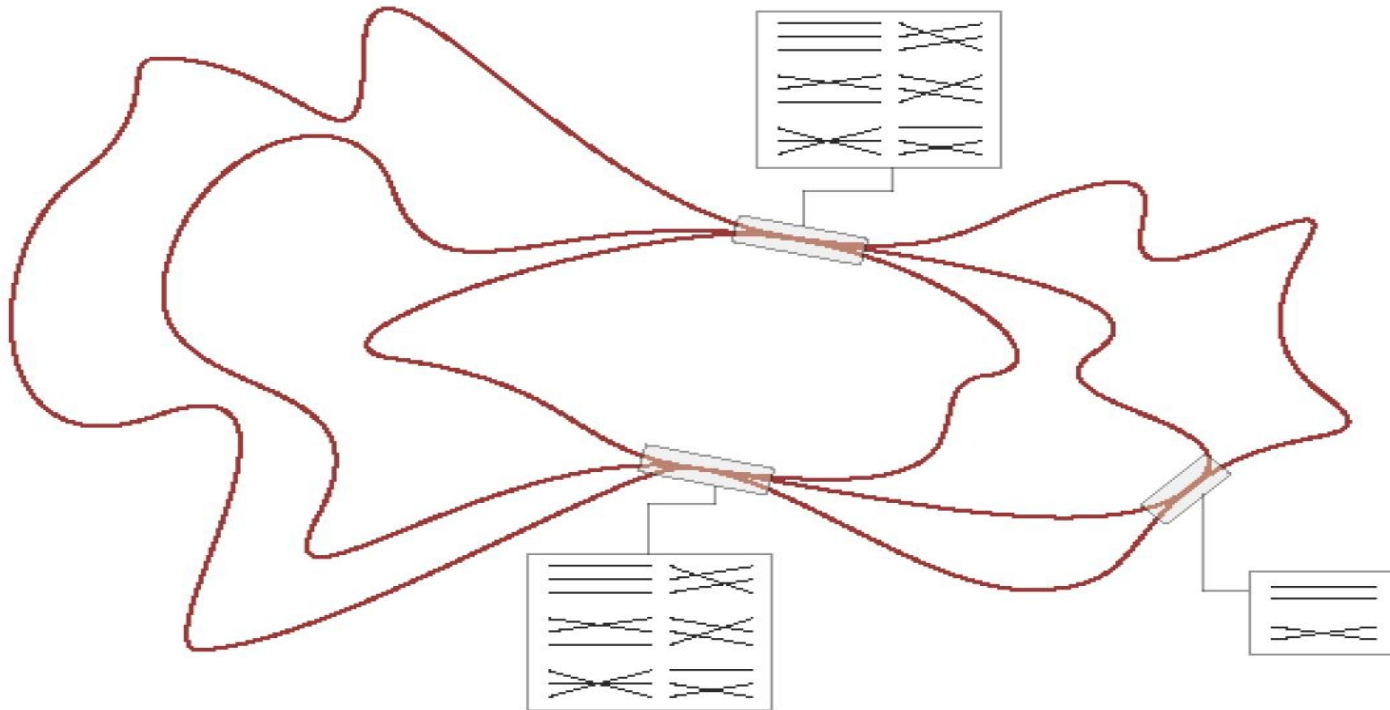
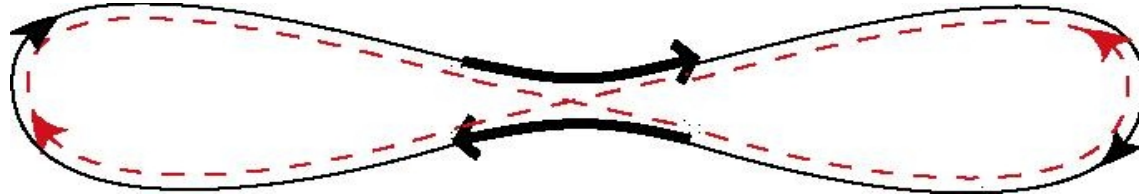
leading term of RMT result

Bunches with encounters

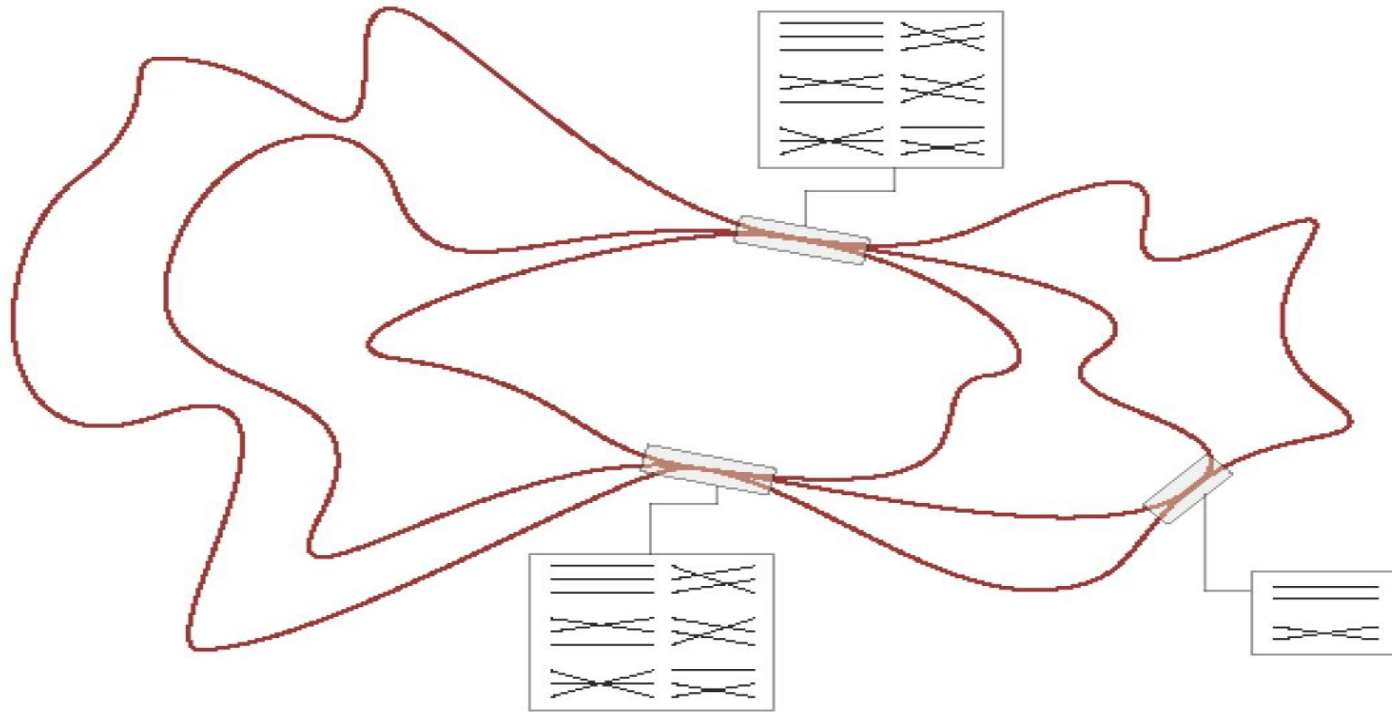
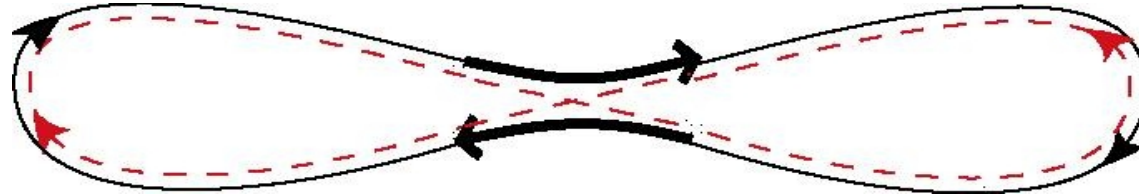
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Full agreement with random matrix theory

Beyond universality

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Beyond universality

Diagonal approximation:

$$Z \sim e^{\pi i(\alpha+\beta-\gamma-\delta)} \frac{\zeta(-i(\alpha+\delta)/\hbar)\zeta(-i(\gamma+\beta)/\hbar)}{\zeta(-i(\alpha+\beta)/\hbar)\zeta(-i(\gamma+\delta)/\hbar)} + \{\gamma \rightarrow -\delta, \gamma \rightarrow -\delta\}$$

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agreement with Bogomolny & Keating

- **finite \hbar**

- **long wires** $\zeta(s) \propto \sqrt{s} \sinh\left(\text{const} \sqrt{s} \frac{\text{diffusion time}}{\text{Heisenberg time}}\right)$

Conclusions

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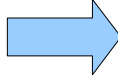

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- Semiclassical approximation:
use generating function, Riemann/Siegel
- Analogy to sigma model

Deviations from universality accessible as well.

Conditions

- **hyperbolicity**  existence of bunches
- **mixing**  universal result
- **semiclassical limit**
quantum time scales \gg classical time scales
- **no other orbit correlations**