# Matrix elements for the quantum cat map: fluctuations in short windows

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# Classical dynamics

- Classical system: geodesic flow on compact (or finite volume) Riemannian manifold $X$.
- Examples to have in mind:
  - Billiards on a planar domain $X$.
  - Modular surface $X = SL_2(\mathbf{Z})\backslash\mathbf{H}$. (Constant negative curvature, chaotic.) (Ignore cont. spectrum in noncompact case.)
- Phase space: $S^*X$.
- Observable: $f(p,q) \in C^\infty(S^*X)$.

# Quantization

- Hilbert space of states: $L^2(X)$. $\psi \in L^2(X)$ called wavefunction; will assume that $\|\psi\|_2 = 1$.
- Quantized Hamiltonian: $\hat{H} = -\Delta$, acting on $L^2(X)$.
- Stationary states:

$$\hat{H}\psi_\lambda = \lambda \cdot \psi_\lambda$$

- Quantized observable: given $f(p,q) \in C^\infty(S^*X)$ define operator $Op(f)$ using PDO's.
- Interested in expectation value $\langle Op(f)\psi, \psi\rangle$ of $f$ w.r.t. state $\psi$. If $f(p,q) = g(q)$ then

$$\langle Op(g)\psi, \psi\rangle = \int_X g(q)|\psi(q)|^2 dq$$

# Quantum expectations

Quantum expectation = classical expectation: as $E \to \infty$,

$$\frac{1}{N(E)} \sum_{\lambda_j \le E} \langle Op(f)\psi_j, \psi_j \rangle \sim \int_{S^*X} f$$

Ergodicity gives that for most $\lambda_j$, $\langle Op(f)\psi_j, \psi_j \rangle \sim \int_{S^*X} f$:

Theorem (Schnirelman, Zelditch, Colin de Verdiere)

*If flow is ergodic then for all observables $f$,*

$$\frac{1}{N(E)} \sum_{\lambda_j \le E} \left| \langle Op(f)\psi_j, \psi_j \rangle - \int_{S^*X} f \right|^2 \to 0$$

*as $E \to \infty$.*

Consequence: There is density one subsequence of eigenfunctions so that for all nice $A \subset X$,

$$\int_A |\psi_j(x)|^2 dx \to \mathrm{vol}(A)/\mathrm{vol}(X)$$

# Quantum variance

**From now on** assume that $\int_{S^*X} f \, d\mu = 0$. Classical ergodicity then gives

$$\frac{1}{N(E)} \sum_{\lambda_j \leq E} |\langle Op(f)\psi_j, \psi_j \rangle|^2 \to 0.$$

More precise rate: Feingold-Peres (1986) prediction for "quantum variance" (for strongly chaotic systems):

$$\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle Op(f)\psi_j, \psi_j \rangle|^2 \sim V_{generic}(f) \cdot N(\lambda)^{-1/2}$$

where

$$V_{generic}(f) := \lim_{T\to\infty} \int_{S^*X} \left| \frac{1}{\sqrt{T}} \int_0^T f_t(x,\xi)\, dt \right|^2 d\mu$$
$$= \int_{-\infty}^{\infty} \langle f_t, f \rangle_{S^*X} \, dt$$

is "classical variance" of $f$ along geodesic flow.

# Distribution of matrix elements

What about the distribution of the normalized fluctuations?
Eckhardt-Fishman-Keating-Agam-Main-Müller (1995): improved on arguments for variance prediction, and numerically found that the distribution of the normalized fluctuations should be *Gaussian*.
Additional numerics:

- Bäcker-Schubert-Stifter 1998 (billiards).
- Barnett 2004, 2006 (billiards).

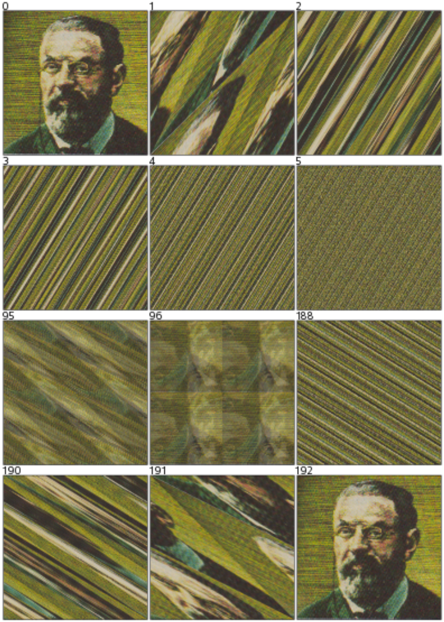
Some theoretical results on variance/fluctuations:

- Modular domain: Luo-Sarnak (2004).
- Cat maps: K.-Rudnick (2002), Kelmer (2005).

Problem: when variance can be determined rigorously, get "wrong" answer. More on this later.

# Cat maps

- Discrete time dynamical system, “toy model” for understanding Quantum Chaos.
- Phase space: Two dimensional torus, $T^2 = \mathbf{R}/\mathbf{Z} \times \mathbf{R}/\mathbf{Z}$.
- Dynamics given by $A \in SL_2(\mathbf{Z})$ acting on $T^2$.
- If $|\operatorname{trace}(A)| > 2$ get classical chaos.

Illustration: 

# Quantization

- Planck's constant: $h = 1/N$, $N$ integer tending to $\infty$.
- State space: $\mathcal{H}_N = L^2(\mathbf{Z}/N\mathbf{Z})$. (Finite dimensional!)
- Observables: Given $f \in C^\infty(T^2)$, $Op_N(f)$ is just some $N \times N$ matrix.
- Quantum propagator: Unitary map $U_N(A)$ acting on state space $L^2(\mathbf{Z}/N\mathbf{Z})$.
- Think of $U_N(A)$ as $e^{-i/h\widehat{H}}$.

# Behaviour of matrix elements

Stationary states: eigenfunctions of propagator

$$U_N(A)\psi_j = \lambda_j \psi_j, \quad i = 1, \dots, N$$

Study (diagonal) matrix elements of observables w.r.t. this basis:

$$\langle \mathrm{Op}_N(f)\psi_j, \psi_j \rangle$$

Quantum variance for cat maps:

**Theorem (Zelditch, 1997)**

*As $N \to \infty$, and $\{\psi_j\}$ an eigenbasis of $L^2(\mathbf{Z}/N\mathbf{Z})$,*

$$\frac{1}{N} \sum_{j=1}^{N} |\langle \mathrm{Op}_N(f)\psi_j, \psi_j \rangle|^2 \to 0$$

# Huge spectral degeneracies

Quantum propagator $U_N(A)$ can have large spectral degeneracies, as big as $N/\log N$.

**Theorem (Schubert)**

*There exists subsequence of $N$'s and $\{\psi_j\}$'s such that*

$$\frac{1}{N}\sum_{j=1}^{N} |\langle \mathrm{Op}_N(f)\psi_j, \psi_j\rangle|^2 \sim \frac{1}{\log N}$$

Remark: shows that analog of Zelditch' upper bound (for geodesic flow on negatively curved manifold) is sharp for quantum maps.

**Theorem (De Biévre, Faure, Nonnenmacher, 2003)**

*If degeneracies are maximal, "scarring" can occur. In particular, there exists sequence $N, \psi_N$ for which $\langle \mathrm{Op}_N(f)\psi_N, \psi_N\rangle \not\to 0$.*

# Getting rid of degeneracies - Hecke operators

Theorem (K.-Rudnick, 2000)

*There exists commuting family $\mathcal{C}_A(N)$ of unitary operators acting on $\mathcal{H}_N$ that commute with $U_N(A)$*

Define *Hecke eigenfunctions* as joint eigenfunctions of all $U \in \mathcal{C}_A(N)$

Still degeneracies, but much smaller ($\ll \log\log N$.)

Theorem (K.-Rudnick, 2000)

*For Hecke eigenfunctions,*

$$\langle \mathrm{Op}_N(f)\psi_j, \psi_j \rangle \ll N^{-1/4+\epsilon}$$

*hence "arithmetic QUE" holds for cat maps.*

# Moments

Theorem (K.-Rudnick, 2005)

*For $N$ prime and $\{\psi_j\}$ a Hecke eigenbasis,*

$$\frac{1}{N}\sum_{j=1}^{N} |\langle \mathrm{Op}_N(f)\psi_j, \psi_j\rangle|^2 = \frac{V_{Hecke}(f)}{N} + o(1/N),$$

$$\frac{1}{N}\sum_{j=1}^{N} |\langle \mathrm{Op}_N(f)\psi_j, \psi_j\rangle|^4 = \frac{C_4(f)}{N^2} + o(1/N^2)$$

Caveat: $V_{Hecke} \neq V_{generic}$.
Also note: $L^4$-result shows that the distribution of the fluctuations **cannot** be Gaussian.

# $L^\infty$-result

If all terms of same size, get *individual* rate of decay:

$$\langle \mathrm{Op}_N(f)\psi_j, \psi_j\rangle \ll N^{-1/2}.$$

Known for primes:

Theorem (Degli Esposti, Graffi, Isola)

*If $N$ is split prime, then*

$$\langle \mathrm{Op}_N(f)\psi_j, \psi_j\rangle \ll_f N^{-1/2}.$$

Theorem (Gurevich, Hadani)

*If $N$ is inert prime, then*

$$\langle \mathrm{Op}_N(f)\psi_j, \psi_j\rangle \ll_f N^{-1/2}.$$

BUT: Kelmer recently proved slower decay for $N =$ prime power.

# Why the wrong variance?

- Canonical basis of observables given by exponentials (indexed by $n \in \mathbf{Z}^2$)
$$g_n(x,y) := e^{2\pi i(xn_1+yn_2)}$$
Turns out that if $m = nA^k \in \mathbf{Z}^2$, then
$$\langle \mathrm{Op}_N(g_n)\psi_j, \psi_j\rangle = \langle \mathrm{Op}_N(g_m)\psi_j, \psi_j\rangle\,, \quad \forall j$$
- Matrix elements corresponding to Fourier coefficients in same $A$-orbit are highly dependent. (To be expected by Egorov!)
- More surprising: Given $A$, can define a binary quadratic form $Q$ with following property:
If $Q(m) = Q(n)$ for $m, n \in \mathbf{Z}^2$, then
$$\langle \mathrm{Op}_N(g_n)\psi_j, \psi_j\rangle = \pm \langle \mathrm{Op}_N(g_m)\psi_j, \psi_j\rangle\,, \quad \forall j$$

# Why the wrong variance, cont'd

- Expect variance of diagonal matrix elements to be:

$$V_{\text{generic}} = \sum_{t=-\infty}^{\infty} \int_{\mathbf{T}^2} f(x)\overline{f(A^t x)}dx.$$

- In terms of Fourier expansion, get:

$$V_{\text{generic}} = \sum_{t=-\infty}^{\infty} \sum_{n \in \mathbf{Z}^2 - \{0\}} \hat{f}(n)\overline{\hat{f}(nA^t)}$$

$$= \sum_{m \in (\mathbf{Z}^2 - \{0\})/\langle A \rangle} \left| \sum_{n \in m\langle A \rangle} \hat{f}(n) \right|^2$$

- Now, $(n_1, n_2) = (m_1, m_2)A^k \Rightarrow (-1)^{n_1 n_2} = (-1)^{m_1 m_2}$, so

$$V_{\text{generic}} = \sum_{m \in (\mathbf{Z}^2 - \{0\})/\langle A \rangle} \left| \sum_{n \in m\langle A \rangle} (-1)^{n_1 n_2} \hat{f}(n) \right|^2$$

- For cat map:

$$V_{\text{Hecke}} := \sum_{\substack{0 \neq m, n \in \mathbf{Z}^2 \\ Q(m) = Q(n)}} (-1)^{n_1 n_2 + m_1 m_2} \widehat{f}(m) \overline{\widehat{f}(n)}$$

$$= \sum_{k \neq 0} \left| \sum_{n \in \mathbf{Z}^2 : Q(n) = k} (-1)^{n_1 n_2} \hat{f}(n) \right|^2$$

- Answers would agree if $A$ acts transitively on hyperbolas $\{n \in \mathbf{Z}^2 : Q(n) = k\}$.
- However: $m = nA^k \Rightarrow Q(m) = Q(n)$, but converse is not true.

# Local independence of matrix elements

Study *local independence* via cancellation in sums over short windows (here $\lambda_j = e^{2\pi i\theta_j}$)

$$\sum_{j:|\theta_j-\theta|<1/L} \langle \mathrm{Op}_N(f)\psi_j, \psi_j\rangle$$

Smoothing: given smooth $h$ with $\int_{-\infty}^{\infty} h(x)^2 dx = 1$, put $h_L(\theta) = \sum_{k\in\mathbf{Z}} h(L(\theta - k))$ and define

$$P(\theta) := \sum_{j=1}^{N} h_L(\theta - \theta_j)\langle \mathrm{Op}_N(f)\psi_j, \psi_j\rangle \sim \sum_{j:|\theta_j-\theta|<1/L} \langle \mathrm{Op}_N(f)\psi_j, \psi_j\rangle$$

Note: $P$ does **not** depend on choice of basis.
Mean value: since we assume $\int_{\mathbf{T}^2} f = 0$, have $\int_0^1 P(\theta)\, d\theta = 0$.
Detect cancellation by looking at variance

$$\mathrm{Var}(P) := \int_0^1 P(\theta)^2\, d\theta$$

## Theorem (K.-Rosenzweig-Rudnick)

*Let $f$ be smooth observable with zero mean. As $N \to \infty$ along primes such that* $\operatorname{ord}(A, N)/\sqrt{N} \to \infty$ *and* $2\operatorname{ord}(A, N) > L \gg N^{1/2+\epsilon}$*, then*

$$\operatorname{Var}(P) = \frac{V_{Hecke}(f)}{L} + O(\sqrt{N}/L^2) \sim \frac{V_{Hecke}(f)}{L}$$

Hence get square root cancellation in short windows; matrix elements behave "independently".

For cat maps, eigenvalues equally spaced, so if $L > \operatorname{ord}(A, N)$

$$\operatorname{Var}(P) = \frac{1}{L} \sum_{k=1}^{\operatorname{ord}(A,N)} \left| \sum_{j:\theta_j = k/\operatorname{ord}(A,N)} \langle \operatorname{Op}_N(f)\psi_j, \psi_j\rangle \right|^2$$

Also, if $\operatorname{ord}(A, N)$ is maximal and $L > N + 1$, get variance for individual matrix elts

$$\operatorname{Var}(P) = \frac{1}{L} \sum_{j=1}^{N} |\langle \operatorname{Op}_N(f)\psi_j, \psi_j\rangle|^2 + o(1/L)$$

# Sketch of proof

With $\Gamma(t) = \sum_{\tau \equiv t \mod \operatorname{ord}(A,N)} \hat{h}(t/L)^2$, Fourier analysis gives

$$\operatorname{Var}(P) \sim \frac{1}{L^2} \sum_{t=1}^{\operatorname{ord}(A,N)-1} \Gamma(t) \left| \operatorname{tr}\left\{ \operatorname{Op}_N(f) U_N(A)^{-t} \right\} \right|^2$$

This, with the trace formula

$$\operatorname{tr}\left\{ \operatorname{Op}_N(f) U_N(A)^{-t} \right\} = \sum_{k \in \mathbf{Z}^2} (-1)^{k_1 k_2} \hat{f}(k) e\left( \frac{q(k, A^t)}{N} \right),$$

where $q(k, A^t)$ is a quadratic form in $k$, gives

$$\operatorname{Var}(P) \sim \frac{1}{L^2} \sum_{k,k' \in \mathbf{Z}^2} (-1)^{k_1 k_2 + k_1' k_2'} \hat{f}(k) \overline{\hat{f}(k')} S(k, k')$$

where

$$S(k, k') = \sum_{t=1}^{\operatorname{ord}(A,N)-1} \Gamma(t) e\left( \frac{q(k, A^t) - q(k', A^t)}{N} \right)$$

# Main term comes from "diagonal"

Diagonal contribution: $q(k, A^t) \equiv q(k', A^t) \mod N$ iff $Q(k) \equiv Q(k') \mod N$.

Easy to see that

$$\sum_{t=1}^{\mathrm{ord}(A,N)-1} \Gamma(t) \sim L$$

and hence main term equals

$$\frac{1}{L^2} \sum_{k,k' \in \mathbf{Z}^2 : Q(k) \equiv Q(k') \mod N} (-1)^{k_1 k_2 + k_1' k_2'} \hat{f}(k) \overline{\hat{f}(k')} \cdot L = V_{\mathrm{Hecke}}(f)/L$$

as promised.

# Off-diagonal terms

For $q(k, A^t) \neq q(k', A^t)$, must bound

$$S(k, k') = \sum_{t=1}^{\mathrm{ord}(A,N)-1} \Gamma(t) e\left(\frac{q(k, A^t) - q(k', A^t)}{N}\right)$$

By (multiplicative) Fourier analysis, enough to bound "Fourier coefficients"

$$E_A(j) := \sum_{t=1}^{\mathrm{ord}(A,N)-1} e\left(\frac{jt}{\mathrm{ord}(A, N)}\right) e\left(\frac{q(k, A^t) - q(k', A^t)}{N}\right)$$

By Polya-Vinogradov, enough to bound "complete" sums,

$$E(\chi) := \sum_{B \in \mathcal{C}_A(N)} \chi(B) e\left(\frac{q(k, B) - q(k', B)}{N}\right)$$

# Bounding the complete exponential sums

Using the full power of Deligne II, Gurevich-Hadani has shown that

$$E(\chi) \ll \sqrt{N}$$

Alternatively, can use RH for curves (Weil) to show

$$E(\chi) \ll \sqrt{N}$$

Upshot:

$$S(k, k') \ll \sqrt{N}$$

so by rapid decay of $\hat{f}(k)$ get that off-diagonal contribution

$$\frac{1}{L^2} \sum_{k,k' \in \mathbf{Z}^2 : Q(k) \not\equiv Q(k') \mod N} (-1)^{k_1 k_2 + k_1' k_2'} \hat{f}(k) \overline{\hat{f}(k')} S(k, k') \ll \frac{\sqrt{N}}{L^2}$$

QED

# Related results for the modular domain

Theorem (Luo-Sarnak, 2005)

*For $f$ a Hecke-Maass form on $\Gamma\backslash\mathbf{H}$, have*

$$V(f) = V_{generic}(f) \cdot L(1/2, f)$$

Fluctuations/moments: formula by Watson (2002) gives

$$|\langle Op(f)\psi, \psi\rangle|^2 \sim L(1/2, Sym^2(\psi) \times f)$$

BUT: moments of $L(1/2, ...)$ expected to blow up!
E.g., Rudnick-Soundararajan (2006) proved lower bounds for moments over families of orthogonal/sympletic holomorphic Hecke forms.

# The distribution of fluctuations for catmaps

Consider normalized fluctuations:

$$F_j^{(N)} := \sqrt{N} \cdot (\langle \mathrm{Op}_N(f)\psi_j, \psi_j \rangle)$$

Group Fourier coefficients on same level set of $Q$ together:

$$f^{\#}(k) := \sum_{\substack{n=(n_1,n_2)\in \mathbf{Z}^2 \\ Q(n)=k}} (-1)^{n_1 n_2} \hat{f}(n)$$

## Conjecture (K.-Rudnick)

*As $N \to \infty$ through primes, the limiting distribution of the normalized matrix elements $F_j^{(N)}$ is that of the random variable*

$$X_f := \sum_{k \neq 0} f^{\#}(k) \, \mathrm{tr}(U_k)$$

*where $\{U_k\}$ are independent random matrices in $SU(2)$, w.r.t. Haar measure.*

# Evidence:

Numerics. (Confirms independence.)
A few moments agrees with conjecture. As $N \to \infty$ through primes,

$$\frac{1}{N}\sum_{j=1}^{N}(F_j^{(N)})^k \to \mathbf{E}(X_f)^k$$

for $k = 2, 4$ (K.-Rudnick, 05). Recently Rosenzweig extended this to $k = 3, 5$ (The $k = 3$ result also holds for short windows.)

For $f(x, y) = e^{2\pi i(xn_1 + yn_2)}$, matrix elements are given by one-variable character sums, and can show

$$|F_j^{(N)}| \leq 2 + O(1/N)$$

In particular: know that distribution has compact support - not Gaussian! Natural guess: Sato Tate distribution, aka semicircle law.

# Fluctuations in windows

Take $f(x,y) = e^{2\pi i(xn_1+yn_2)}$.

Expected behaviour

- For tiny window (only one eigenvalue): should get Sato-Tate.
- For longer window: should get sum of independent Sato-Tate's, hence Gaussian behaviour.

Nicely illustrated by numerics due to Rosenzweig: