

Structure and enumeration of two-connected graphs with prescribed three-connected components

Andrei Gagarin*, Gilbert Labelle^{†‡}; Pierre Leroux^{†‡§}; Timothy Walsh^{†‡}

Abstract

We adapt the classical 3-decomposition of any 2-connected graph to the case of simple graphs (no loops or multiple edges). By analogy with the block-cutpoint tree of a connected graph, we deduce from this decomposition a bicolored tree $tc(g)$ associated with any 2-connected graph g , whose white vertices are the 3-components of g (3-connected components or polygons) and black vertices are *bonds* arising from separating pairs of vertices of g and linking together the 3-components. Two fundamental species relationships on graphs and networks follow from this construction. The first one is a dissymmetry theorem which leads to the expression of the class $\mathcal{B} = \mathcal{B}(\mathcal{F})$ of 2-connected graphs, all of whose 3-connected components belong to a given class \mathcal{F} of 3-connected graphs, in terms of various rootings of \mathcal{B} . The second one is a functional equation which characterizes the corresponding class $\mathcal{R} = \mathcal{R}(\mathcal{F})$ of two-pole networks all of whose 3-connected components are in \mathcal{F} . All the rootings of \mathcal{B} are then expressed in terms of \mathcal{F} and \mathcal{R} . There follow corresponding identities for all the associated power series, in particular, the edge index series. These results are expressed in terms of species of structures. Using these results we enumerate several classes of labelled and unlabelled graphs, including series-parallel graphs, 2-connected planar graphs, $K_{3,3}$ -free 2-connected graphs and $K_{3,3}$ -free projective planar and toroidal graphs.

MSC: 05C30; 05A15; 05C05; 05C40; 05C75

Keywords: 3-Connected decomposition; Dissymmetry theorem; 2-Trees; Enumeration; Combinatorial species

1 Introduction

A graph is assumed to be simple, that is, with no loops or multiple edges. A graph G is called *k-connected* if at least k of its vertices and their incident edges must be deleted to

*Jodrey School of Computer Science, Acadia University, Wolfville, Nova Scotia, Canada, B4P 2R6.

[†]Laboratoire de Combinatoire et d'Informatique Mathématique (LaCIM), Université du Québec à Montréal (UQAM), Montréal, Québec, CANADA, H3C 3P8.

[‡]With the partial support of NSERC (Canada).

[§]Regretfully, Pierre Leroux passed away soon after submitting this paper.

[¶]*E-mail addresses:* andrei.gagarin@acadiau.ca (A. Gagarin), labelle.gilbert@uqam.ca (G. Labelle), walsh.timothy@uqam.ca (T.R. Walsh).

disconnect it. In general, a k -connected graph is assumed to have more than k vertices. However, by convention, the complete graph K_2 will be considered as a 2-connected graph. A connected graph G is *planar* if there exists a 2-cell embedding (i.e. each face is homeomorphic to an open disk) of G on the sphere, with similar definitions for *toroidal* and *projective-planar* graphs.

A *species* is a class \mathcal{C} of labelled combinatorial structures (for example, graphs or rooted trees) which is closed under isomorphism. Each \mathcal{C} -structure has an underlying set (for example, the vertex set of a graph), and isomorphisms are obtained by relabelling along bijections between the underlying sets. Unlabelled structures are defined as isomorphism classes of structures. We sometimes denote a species by the name of representatives of the isomorphism classes. For example, K_n is used to denote the species of complete graphs on n vertices. One advantage of species is that very often combinatorial identities can be expressed at this structural level, within the algebra of species and their operations (sum, product, substitution and a special substitution, of networks for edges, etc.). There follow corresponding identities for the various generating series that are used for labelled and/or unlabelled enumeration. The theory of combinatorial species was introduced by Joyal in [11]. The reader is referred to the book [1] for more details on species, their operations and associated series.

The two-pole networks that we use have distinguished poles 0 and 1 and are called 01-networks. A 01-*network* (or more simply a *network*) is defined as a connected graph N with two distinguished vertices 0 and 1, such that the graph $N \cup 01$ is 2-connected, where the notation $N \cup ab$ is used for the graph obtained from N by adding the edge ab if it is absent. See Figure 1 for an example. Vertices 0 and 1 are called the *poles* of N , and all the other vertices of N are called *internal* vertices. The internal vertices of a network form its underlying set. A trivial network, consisting of only the poles 0 and 1 and having no edges, is denoted by $\mathbb{1}$.

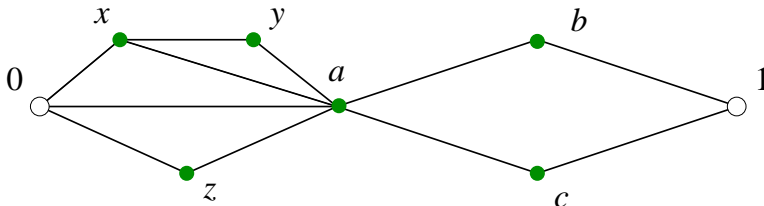


Figure 1: A 01-network.

In this paper, we first adapt to the case of simple graphs the classical 3-decomposition of 2-connected multigraphs (see MacLane [14] Tutte [21, 22], Hopcroft and Tarjan [10] and Cunningham and Edmonds [4]). By analogy with the block-cutpoint tree $bc(c)$ of a connected graph c , we deduce from this decomposition a bicolored tree $tc(g)$ associated with any 2-connected graph g , whose white vertices are the *3-components* of g (3-connected components or polygons) and black vertices are *bonds* arising from separating pairs of vertices of g , linking together these 3-components and acting as hinges.

Two fundamental relationships on graphs and networks follow from this construction. The first one is a dissymmetry theorem which leads to the expression of a class $\mathcal{B} = \mathcal{B}(\mathcal{F})$ of 2-connected graphs, all of whose 3-connected components belong to a given class \mathcal{F}

of 3-connected graphs, in terms of various rootings of \mathcal{B} , where a graph in \mathcal{B} is *rooted* by distinguishing either one of its 3-components, or a pair of vertices that are shared by several 3-components and, possibly, also one of these 3-components. See Theorem 1 below. The second one is a functional equation which characterizes the corresponding class $\mathcal{R} = \mathcal{R}(\mathcal{F})$ of non-trivial 01-networks all of whose 3-connected components are in \mathcal{F} . See Theorem 8. Note that the 3-components of a network N are obtained by considering the 3-decomposition of the graph $N \cup 01$.

All the rootings of \mathcal{B} are then expressible in terms of \mathcal{F} and \mathcal{R} and hence also \mathcal{B} itself by virtue of the dissymmetry theorem. Although more or less implicit in the previous work of one of the authors (see [24, 25]), these identities are given here for the first time in the structural context of species. There follow corresponding identities for all the associated series, in particular, the edge index series, and numerous enumerative consequences are obtained.

Among the examples that we have in mind and that will be discussed further in this paper are the following:

1. If we take $\mathcal{F} = \mathcal{F}_{\text{all}}$, the class of all 3-connected graphs, then we have $\mathcal{B}(\mathcal{F}) = \mathcal{B}_{\text{all}}$, the class of all 2-connected graphs, and $\mathcal{R}(\mathcal{F}) = \mathcal{R}_{\text{all}}$, the class of all non-trivial 01-networks.
2. If we take $\mathcal{F} = 0$, the empty species, then \mathcal{B} and \mathcal{R} are the classes of 2-connected *series-parallel* graphs and *series-parallel* networks, respectively.
3. One of the motivations for the present paper was to extend some earlier tables for the number of $K_{3,3}$ -free projective planar and toroidal 2-connected graphs (see [8]), which require enumeration of *strongly planar networks*, that is non-trivial networks N such that the graph $N \cup 01$ is planar. This class, denoted by \mathcal{N}_P , can be obtained by considering the class $\mathcal{F} = \mathcal{F}_P$ of planar 3-connected graphs. Then the corresponding species $\mathcal{B}_P = \mathcal{B}(\mathcal{F}_P)$ is the class of planar 2-connected graphs, since a graph is planar if and only if all its 3-connected components are planar, and $\mathcal{R}(\mathcal{F}_P) = \mathcal{N}_P$ is the class of strongly planar networks.
4. As quoted by Thomas ([19], p. 1, Theorem 1.2), Wagner [23] has shown that a 2-connected graph is $K_{3,3}$ -free if and only if it can be obtained from planar graphs and K_5 's by means of 2-sums (see also Kelmans [12]). This means that if we take $\mathcal{F} = \mathcal{F}_P + K_5$, the corresponding species $\mathcal{B} = \mathcal{B}(\mathcal{F})$ is the class of $K_{3,3}$ -free 2-connected graphs. This fact was also observed by Giménez, Noy and Rué in [9].

Section 2 contains the dissymmetry theorem. Section 3 discusses various operations on 01-networks, in particular, series and parallel composition and the substitution of networks for edges in graphs or networks. It also presents the fundamental relationship characterizing the species $\mathcal{R} = \mathcal{R}(\mathcal{F})$ and the expressions of the various rootings of \mathcal{B} in terms of \mathcal{F} and \mathcal{R} . Applications to the labelled enumeration of these species are also presented. Section 4 is devoted to the series techniques for species of graphs and networks that are necessary for their unlabelled enumeration. These results are then applied to the enumeration of several classes of graphs and networks in Section 5.

2 A dissymmetry theorem for 2-connected graphs

In the standard decomposition of a 2-connected multigraph (multiple edges allowed but no loops) into 3-components (see [21, 22, 10] and [4]), the components are either 3-connected graphs (called here *3-connected components*), polygons with at least three sides, or bonds, which are sets of at least 3 multiple edges.

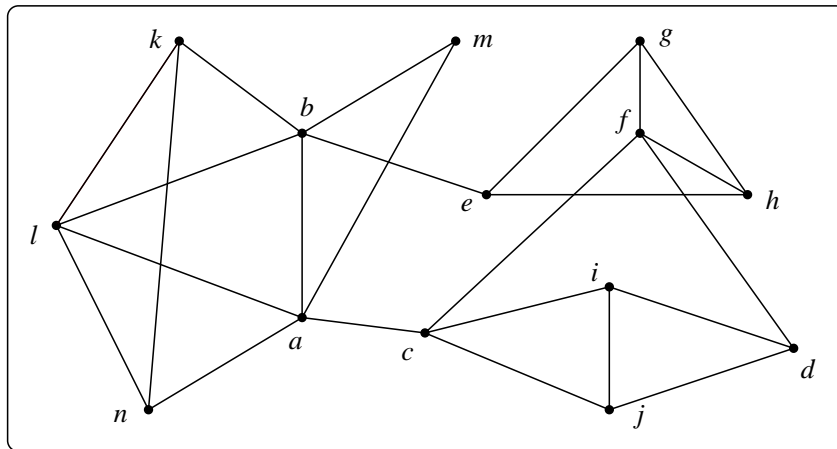


Figure 2: A 2-connected graph g .

However, in case of simple graphs, bonds are not needed as 3-components and the decomposition is simpler. We illustrate the construction with the graph g of Figure 2. By definition, a *separating pair* of a 2-connected graph g is a pair of vertices $\{x, y\}$ whose removal disconnects the graph. Then one can re-introduce the two vertices x and y together with their incident edges and the edge xy into each resulting connected component.

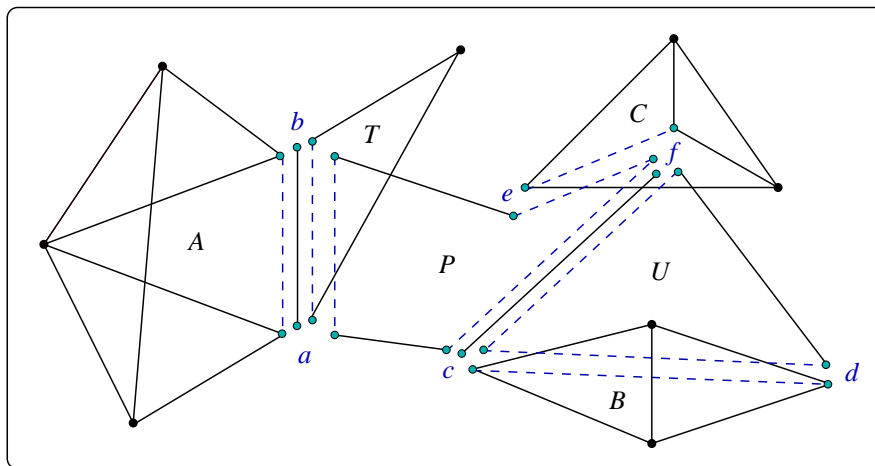


Figure 3: The 3-decomposition of g .

We also keep a record whether the edge xy is present or not in the original graph g . This has been done in Figure 3 for the separating pairs $\{a, b\}$, $\{c, d\}$, $\{c, f\}$ and $\{e, f\}$ and

we see the 3-components appearing: the 3-connected components A , B , and C , and the polygons P , T and U . The above dissection could also be performed for the separating pairs $\{a, e\}$ or $\{c, e\}$, for example, but that would cut the polygon P into smaller polygons; so it is not done since the maximality of the polygonal components ensures the unicity of the decomposition. We refer the reader to the bibliography for more details. The essential separating pairs of g , ($\{a, b\}$, $\{c, d\}$, $\{c, f\}$ and $\{e, f\}$ in the example) will be referred to as the *bonds* of the 3-decomposition. Hence a bond $\{x, y\}$ links together two or more 3-components and, possibly, the edge xy , with the exception of two polygons alone which is forbidden.

By analogy with the block-cutpoint tree $bc(c)$ of a connected graph c , we deduce from this decomposition a bicolored tree $tc(g)$ associated with any 2-connected graph g , whose white vertices are the 3-components of g (3-connected components or polygons) and whose black vertices are the separating pairs linking together these 3-components (the bonds). See Figure 4.

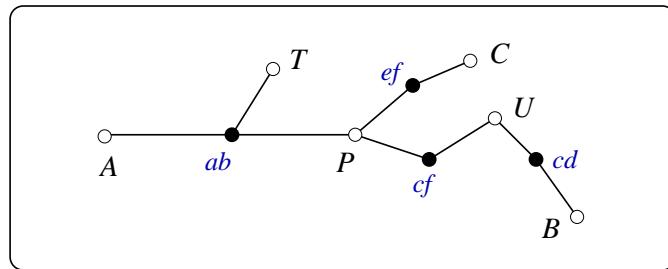


Figure 4: The tc -tree of g .

Now let \mathcal{F} be a given species of 3-connected graphs and $\mathcal{B} = \mathcal{B}_{\mathcal{F}}$ be the class of 2-connected graphs all of whose 3-connected components are in \mathcal{F} . Notice that, by convention, K_2 is in \mathcal{B} . We introduce the following classes of *rooted* graphs in \mathcal{B} which are relative to the concept of tc -tree:

\mathcal{B}° denotes the class of all graphs $g \in \mathcal{B}$, rooted at a distinguished 3-component (a 3-connected component or a polygon);

\mathcal{B}^{\bullet} denotes the class of all graphs $g \in \mathcal{B}$ together with a distinguished bond;

$\mathcal{B}^{\circ\bullet}$ denotes the class of all graphs $g \in \mathcal{B}$ with a distinguished pair of adjacent 3-component and bond.

The following theorem is one of many generalizations of Otter's dissymmetry theorem for trees [16].

Theorem 1 (Dissymmetry Theorem for 2-connected graphs). *Let \mathcal{F} be a given species of 3-connected graphs and $\mathcal{B} = \mathcal{B}_{\mathcal{F}}$ be the class of 2-connected graphs all of whose 3-connected components are in \mathcal{F} . We then have the following identity (species isomorphism):*

$$\mathcal{B}^{\circ} + \mathcal{B}^{\bullet} = \mathcal{B} - K_2 + \mathcal{B}^{\circ\bullet}. \quad (1)$$

Proof. The proof uses the concept of *center* of a tree. Notice that all the leaves of a tc -tree are of the same color (white). This implies that its center is a vertex, black or white. Now a structure s belonging to the left-hand side of (1) is a graph $g \in \mathcal{B}$ which

is rooted at either a tricomponent or a bond of g , that is at a white or black vertex of $\text{tc}(g)$. It can happen that the rooting is performed right at the center. This is canonically equivalent to doing nothing and is represented by the term $\mathcal{B} - K_2$ in the right-hand side of (1). On the other hand, if the rooting is done at an off-center vertex, black or white, then there is a unique adjacent vertex of the other color in $\text{tc}(g)$ which is located closer to the center, thus defining a unique $\mathcal{B}^{\circ\bullet}$ -structure. It is easily checked that this correspondence is bijective and independent of any labelling, giving the desired species isomorphism. \blacksquare

Our next goal is to find closed form expressions for the species \mathcal{B}° , \mathcal{B}^\bullet and $\mathcal{B}^{\circ\bullet}$. This will be achieved using the operation of substitution of 01-networks for the edges of *core* graphs, as explained in the next section.

3 Operations on networks and their exponential generating functions

We first describe the exponential generating functions which are used for the labelled enumeration of graphs and networks and of related species, according to the number of edges. For a species \mathcal{G} of graphs, the exponential generating function $\mathcal{G}(x, y)$, where the variable y acts as an edge counter, is defined by

$$\mathcal{G}(x, y) = \sum_{n \geq 0} g_n(y) \frac{x^n}{n!} = \sum_{n \geq 0} \sum_{m \geq 0} g_{n,m} y^m \frac{x^n}{n!}, \quad (2)$$

where $g_{n,m}$ is the number of graphs in \mathcal{G} with m edges over a given n -element set of vertices. Similar definitions can be given for associated (for example, rooted) species of graphs. For a species \mathcal{N} of 01-networks, the exponential generating function $\mathcal{N}(x, y)$ is defined by

$$\mathcal{N}(x, y) = \sum_{n \geq 0} \nu_n(y) \frac{x^n}{n!} = \sum_{n \geq 0} \sum_{m \geq 0} \nu_{n,m} y^m \frac{x^n}{n!}, \quad (3)$$

where $\nu_{n,m}$ is the number of 01-networks in \mathcal{N} with m edges over an n -element set of internal vertices.

We define an operator τ acting on 01-networks, $N \mapsto \tau \cdot N$, that interchanges the poles 0 and 1. A class \mathcal{N} of networks is called *symmetric* if $N \in \mathcal{N} \implies \tau \cdot N \in \mathcal{N}$.

If \mathcal{M} is a species of networks not containing the edge 01, then we denote by $y\mathcal{M}$ the class obtained by adding this edge to all the networks in \mathcal{M} . Observe that there are two distinct networks on the empty set, namely, the *trivial network* $\mathbb{1}$ consisting of two isolated vertices 0 and 1, and the one-edge network $y\mathbb{1}$.

Let B be a given species of 2-connected graphs, for example, $B = \mathcal{B}_{\text{all}}$, the class of all 2-connected graphs, $B = K_2$, or $B = \mathcal{B}_P$, the class of all 2-connected planar graphs. We denote by $B^{(y)}$ the species of graphs obtained by selecting and removing an edge from graphs in B in all possible ways. Note that the removed edge is remembered so that

$$B^{(y)}(x, y) = \frac{\partial}{\partial y} B(x, y). \quad (4)$$

If, moreover, the endpoints of the selected and removed edge are stripped of their labels and numbered as 0 and 1, in all possible ways, the resulting class of networks is denoted by $B_{0,1}$. For example, $(\mathcal{B}_{all})_{0,1}$ is the class of networks having non-adjacent poles, $(K_2)_{0,1} = \mathbb{1}$, and the class of strongly planar networks can be expressed as $\mathcal{N}_P = (1 + y)(\mathcal{B}_P)_{0,1} - \mathbb{1}$, where the multiplication $y \cdot B_{0,1}$ corresponds to adding the edge 01 to all networks in $B_{0,1}$. Another example, $(K_5)_{0,1}$, is illustrated in Figure 11 (i). Relabelling the two poles yields the identity

$$x^2 B_{0,1}(x, y) = 2 B^{(y)}(x, y), \quad (5)$$

so that

$$B_{0,1}(x, y) = \frac{2}{x^2} \frac{\partial}{\partial y} B(x, y). \quad (6)$$

3.1 Series composition

Definition 2 a) Let M and N be two non-trivial disjoint networks. The *series composition* of M followed by N , denoted by $M \cdot_s N$, is a vertex-rooted network whose underlying set is the union of the underlying sets of M and N plus an extra element. It is obtained by taking the graph union of M and N , where the 1-pole of M is identified with the 0-pole of N , and this *connecting vertex* is labelled by the extra element and is the root of $M \cdot_s N$. See Figure 5.

b) The network obtained from a series composition by removing the distinction from the root-vertex is called an *s-network*.

c) The *series composition* $\mathcal{M} \cdot_s \mathcal{N}$ of two species of non-trivial networks \mathcal{M} and \mathcal{N} is the class obtained by taking all series compositions $M \cdot_s N$ of networks with $M \in \mathcal{M}$ and $N \in \mathcal{N}$.

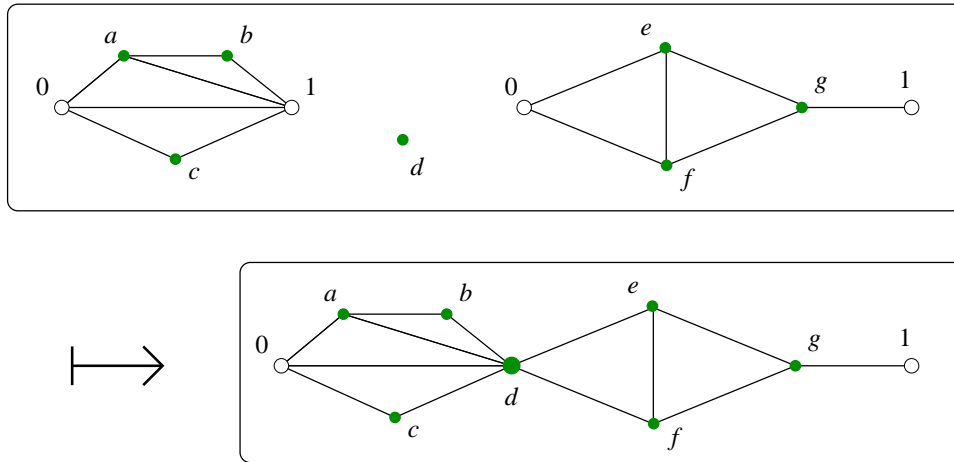


Figure 5: Series composition of networks.

The species $\mathcal{M} \cdot_s \mathcal{N}$ can be expressed as the species product $\mathcal{M} X \mathcal{N}$, where the factor X corresponds to the connecting vertex, and we have

$$(\mathcal{M} \cdot_s \mathcal{N})(x, y) = x \mathcal{M}(x, y) \mathcal{N}(x, y). \quad (7)$$

If for any $\mathcal{M} \cdot_s \mathcal{N}$ -structure the two components $M \in \mathcal{M}$ and $N \in \mathcal{N}$ are uniquely determined by the resulting network (for example, if no $M \in \mathcal{M}$ is an s -network), then the series composition is called *canonical* and the rooting of the connecting vertex can be neglected. We say that a species of networks \mathcal{R} is *closed under series composition and decomposition* if, for any s -network R , R is in \mathcal{R} if and only if each individual factor of R is in \mathcal{R} .

Proposition 3 *Let \mathcal{R} be a species of networks which is closed under series composition and decomposition and let \mathcal{S} denote the class of s -networks in \mathcal{R} . Then we have*

$$\mathcal{S} = (\mathcal{R} - \mathcal{S}) \cdot_s \mathcal{R}, \quad (8)$$

the series composition being canonical, and also

$$\mathcal{S} = \frac{X\mathcal{R}^2}{1 + X\mathcal{R}}. \quad (9)$$

Proof. Any s -network in \mathcal{R} can be decomposed uniquely into a first network which is not an s -network followed by an arbitrary network of \mathcal{R} , whence (8). We also have $\mathcal{S} = (\mathcal{R} - \mathcal{S}) X \mathcal{R}$, and solving for \mathcal{S} yields (9). ■

Corollary 4 *Under the hypothesis of Proposition 3, for the corresponding exponential generating functions, we have*

$$\mathcal{S}(x, y) = \frac{x\mathcal{R}^2(x, y)}{1 + x\mathcal{R}(x, y)}. \quad (10)$$

Remark. Iterating the idea behind the decomposition (8), one can obtain the more symmetric canonical decomposition

$$\mathcal{S} = (\mathcal{R} - \mathcal{S}) \cdot_s (\mathcal{R} - \mathcal{S}) + (\mathcal{R} - \mathcal{S}) \cdot_s (\mathcal{R} - \mathcal{S}) \cdot_s (\mathcal{R} - \mathcal{S}) + \dots. \quad (11)$$

3.2 Parallel composition

Definition 5 a) Let N be a finite set of disjoint non-trivial 01-networks having non-adjacent poles. The *parallel composition* of N is the *partitioned* 01-network obtained by taking the union of the graphs in N , where all 0-poles are merged into one 0-pole, and similarly for the 1-poles; the partition of the internal vertices into those of networks of N is part of the structure. An example is given in Figure 6. By convention, the parallel composition of an empty set of networks is the trivial network $\mathbb{1}$, while the parallel composition of one network is the network itself.

b) The underlying (unpartitioned) network of a parallel composition of at least two networks is called a *p-network*. Any network having adjacent poles is also considered as a *p-network*.

c) If \mathcal{N} is a species of non-trivial 01-networks having non-adjacent poles, then the *parallel composition* of \mathcal{N} is defined as the class of all parallel compositions of finite sets of disjoint networks in \mathcal{N} . This operation is denoted by $E(\mathcal{N})$, the ordinary composition of species, where E denotes the species of sets, since a parallel composition can be seen as an assembly of networks, with all 0-poles and, respectively, all 1-poles identified. As mentioned above, $\mathbb{1} = E_0(\mathcal{N})$ and $\mathcal{N} = E_1(\mathcal{N})$, where E_k is the species of sets of cardinality k , $k = 0, 1$.

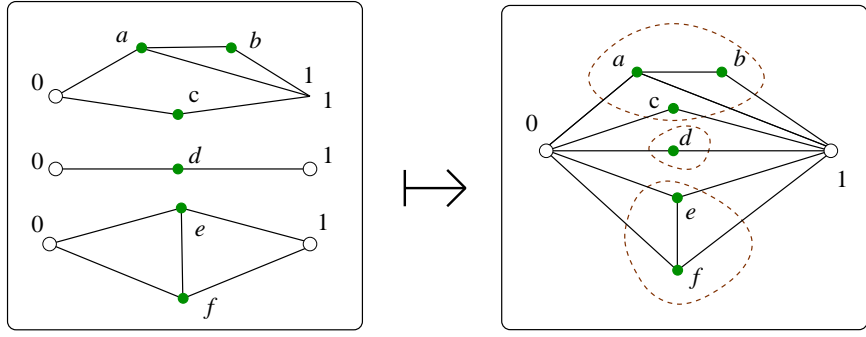


Figure 6: Parallel composition of networks.

If each network in a class \mathcal{M} can be viewed unambiguously as a parallel composition of networks in \mathcal{N} , then we say that the parallel composition is *canonical*. This happens if and only if none of the networks in \mathcal{N} is a p -network. Then we can write $\mathcal{M} = E(\mathcal{N})$, and, for the exponential generating functions, we have

$$\mathcal{M}(x, y) = \exp(\mathcal{N}(x, y)). \quad (12)$$

3.3 The \uparrow -composition (substitution of networks for edges)

Definition 6 Let \mathcal{M} be a species of graphs (or networks) and \mathcal{N} be a symmetric species of non-trivial networks. We denote by $\mathcal{T} = \mathcal{M} \uparrow \mathcal{N}$ the class of pairs (M, T) such that

1. the graph (or network) M (called the *core*) is in \mathcal{M} ,
2. there exists a family $\{N_e\}$ of networks in \mathcal{N} (called the *components*) such that the graph (or network) T can be obtained from M by substituting N_e for each edge e of M , the poles of N_e being identified with the extremities of e .

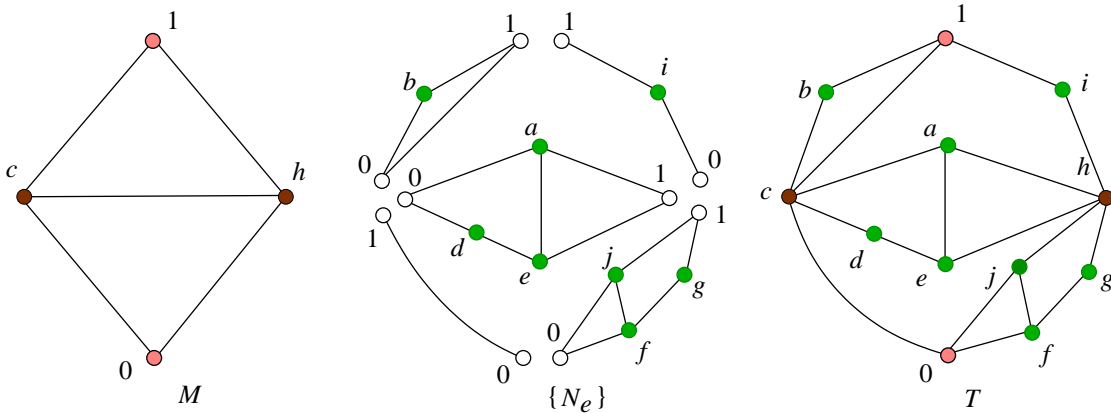


Figure 7: Example of a $(\mathcal{M} \uparrow \mathcal{N})$ -structure (M, T) , with $\mathcal{M} = K_4 \setminus e$.

An example of an $(\mathcal{M} \uparrow \mathcal{N})$ -structure (M, T) is presented in Figure 7, where $\mathcal{M} = K_4 \setminus e$ and \mathcal{N} is the class of all networks. Notice that if a network N_e substituted for an

edge $e = xy$ is not the one-edge network $y\mathbb{1}$, then the pair of vertices $\{x, y\}$ is a separating pair of the resulting graph (or network) T .

As another example, take $\mathcal{G} = K_2$ and let \mathcal{N} be any symmetric species of networks. Then $K_2 \uparrow \mathcal{N}$ consists of rooted graphs obtained from the networks in \mathcal{N} by labelling the two poles 0 and 1, the rooting being at this pair of vertices.

Proposition 7 *Let \mathcal{M} be a species of graphs (or networks) and \mathcal{N} be any symmetric species of non-trivial networks. Then, for the labelled enumeration, we have*

$$(\mathcal{M} \uparrow \mathcal{N})(x, y) = \mathcal{M}(x, \mathcal{N}(x, y)). \quad (13)$$

Proof. See [24] or [6]. ■

The composition $\mathcal{M} \uparrow \mathcal{N}$ is called *canonical* if for any structure $(M, T) \in \mathcal{M} \uparrow \mathcal{N}$ the core $M \in \mathcal{M}$ is uniquely determined by the graph (or network) T . In this case, we can identify $\mathcal{M} \uparrow \mathcal{N}$ with the species of resulting graphs (or networks) T .

Another example of an \uparrow -composition is an h -network, in which the core is a network derived from a 3-connected graph and each of the components is a network derived from a 2-connected graph. A theorem of Trakhtenbrot ([20], see [24] and [25] for a statement of this theorem in English) says that any network with at least two edges (where a network may have parallel edges) is either an s -network, a p -network or an h -network (exclusive or) and that the composition is canonical provided that none of the components of an s -network is an s -network and none of the components of a p -network is a p -network. A proof of this theorem in English is available from the fourth author on request. In Theorem 8 below, this theorem is applied to networks derived from graphs all of whose 3-connected components belong to a given species.

3.4 Functional equations

Let \mathcal{F} be a species of 3-connected graphs; we denote by $\mathcal{B} = \mathcal{B}_{\mathcal{F}}$ (resp., $\mathcal{R} = \mathcal{R}_{\mathcal{F}}$) the class of all 2-connected graphs (resp., non-trivial networks) all of whose 3-connected components are in \mathcal{F} , where the 3-components of a network N are defined by applying the 3-decomposition to the graph $N \cup 01$. Then

$$\mathcal{R} = (1 + y)\mathcal{B}_{0,1} - \mathbb{1}, \quad (14)$$

where multiplication by y corresponds to adding the edge 01 to all networks in $\mathcal{B}_{0,1}$.

One example of a non-canonical \uparrow -composition is given by $\mathcal{F} \uparrow \mathcal{R}$ which represents the species of 2-connected graphs in \mathcal{B} rooted at some 3-connected component. By contrast, as stated in Theorem 8 below, any composition of the form $\mathcal{F}_{0,1} \uparrow \mathcal{N}$ is canonical.

Denote by \mathcal{S} the subclass of \mathcal{R} consisting of s -networks. By virtue of Proposition 3, we have

$$\mathcal{S} = \frac{X\mathcal{R}^2}{1 + X\mathcal{R}}. \quad (15)$$

Parts a) and b) of the following theorem can be seen as a species form of Trakhtenbrot's decomposition theorem restricted to networks with no parallel edges.

Theorem 8 *Let \mathcal{F} be a species of 3-connected graphs and $\mathcal{R} = \mathcal{R}_{\mathcal{F}}$ be the corresponding class of 01-networks associated with \mathcal{F} . Then we have:*

a) *For any (symmetric) species \mathcal{N} of non-trivial networks, the composition $\mathcal{F}_{0,1} \uparrow \mathcal{N}$ is canonical.*

b) *Let \mathcal{H} denote the subclass of h -networks, defined by*

$$\mathcal{H} = \mathcal{F}_{0,1} \uparrow \mathcal{R}. \quad (16)$$

Then we have

$$\mathcal{R} = \mathcal{S} + \mathcal{P} + \mathcal{H}, \quad (17)$$

where \mathcal{P} denotes the subclass of p -networks of \mathcal{R} (see Definition 5 b)), which satisfies

$$\mathcal{P} = (1 + y)E_{\geq 2}(\mathcal{H} + \mathcal{S}) + y(\mathcal{H} + \mathcal{S}) + y\mathbb{1}. \quad (18)$$

c) *The species \mathcal{R} is characterized by the functional relation*

$$\mathcal{R} = (1 + y)E(\mathcal{F}_{0,1} \uparrow \mathcal{R} + \frac{X\mathcal{R}^2}{1 + X\mathcal{R}}) - \mathbb{1}. \quad (19)$$

Proof. As mentioned above, parts a) and b) are essentially a reformulation of Trakhtenbrot's decomposition theorem for networks, where parallel edges are not allowed. Intuitively, for a), observe that in any $\mathcal{F}_{0,1}$ -network M , the poles are non-adjacent. Hence, the same will be true for any network T arising from a $\mathcal{F}_{0,1} \uparrow \mathcal{N}$ -structure (M, T) . Applying the 3-decomposition of 2-connected graphs of Section 2 to the graph $T \cup 01$, we can see that there is a unique 3-connected component in T containing the vertices 0 and 1, namely M itself.

For b), it is easy to see whether a network N having non-adjacent poles is an s -network (the graph is not 2-connected) or a p -network (the poles form a separating pair) and, otherwise, that N is in fact of the form $\mathcal{F}_{0,1} \uparrow \mathcal{R}$, again using the 3-decomposition of 2-connected graphs. If the network has adjacent poles, then it must be of the form $y(\mathbb{1} + \mathcal{H} + \mathcal{S} + E_{\geq 2}(\mathcal{H} + \mathcal{S})) = yE(\mathcal{H} + \mathcal{S})$, so that (17) and (18) are satisfied.

c) Putting (17) and (18) together, we find that

$$\begin{aligned} \mathcal{R} &= \mathcal{H} + \mathcal{S} + y\mathbb{1} + y(\mathcal{H} + \mathcal{S}) + (1 + y)E_{\geq 2}(\mathcal{H} + \mathcal{S}) \\ &= (1 + y)E(\mathcal{H} + \mathcal{S}) - \mathbb{1} \end{aligned} \quad (20)$$

and (19) follows from (20), (16) and (15). ■

We are now in position to express the three rootings \mathcal{B}° , \mathcal{B}^\bullet and $\mathcal{B}^{\circ\bullet}$ of the species $\mathcal{B} = \mathcal{B}_{\mathcal{F}}$ associated with a given species of 3-connected graphs \mathcal{F} , which occur in the Dissymmetry Theorem for 2-connected graphs (Theorem 1), in terms of the corresponding class $\mathcal{R} = \mathcal{R}_{\mathcal{F}}$ of 01-networks. Let's denote by \mathcal{C} the species of polygons, i.e., (unoriented) cycles of length ≥ 3 .

Theorem 9 *We have the following identities:*

$$\mathcal{B}^\circ = \mathcal{F} \uparrow \mathcal{R} + \mathcal{C} \uparrow (\mathcal{R} - \mathcal{S}), \quad (21)$$

$$\mathcal{B}^\bullet = K_2 \uparrow ((1 + y)E_{\geq 2}(\mathcal{H} + \mathcal{S}) - E_2(\mathcal{S})) \quad (22)$$

$$= K_2 \uparrow (\mathcal{R} - (1 + y)(\mathcal{H} + \mathcal{S}) - y\mathbb{1} - E_2(\mathcal{S})), \quad (23)$$

$$\mathcal{B}^{\circ\bullet} = K_2 \uparrow ((\mathcal{H} + \mathcal{S})(\mathcal{R} - y\mathbb{1}) - \mathcal{S}^2), \quad (24)$$

where \mathcal{R} is characterized by equation (19), $\mathcal{S} = \frac{X\mathcal{R}^2}{1+X\mathcal{R}}$ and $\mathcal{H} = \mathcal{F}_{0,1} \uparrow \mathcal{R}$.

Proof. Recall that \mathcal{B}° denotes the class of all graphs $g \in \mathcal{B}$ rooted at a distinguished 3-component. If the distinguished component C is a 3-connected graph, then g is obtained by replacing each edge of C by a network in \mathcal{R} . Otherwise, C is a polygon and g will be obtained by replacing each edge of C by a network in $\mathcal{R} - \mathcal{S}$, by virtue of the maximality of polygonal 3-components. This establishes (21).

Also, recall that the $K_2 \uparrow \mathcal{N}$ -composition transforms networks \mathcal{N} into rooted graphs by labelling the two poles 0 and 1, the rooting being at this pair of vertices. Now, \mathcal{B}^\bullet denotes the class of all graphs $g \in \mathcal{B}$ together with a distinguished bond $\{a, b\}$. This separating pair $\{a, b\}$ decomposes g into two or more pieces which can be seen as either h -networks or s -networks. This yields the term $(1 + y)E_{\geq 2}(\mathcal{H} + \mathcal{S})$, the factor $(1 + y)$ accounting for the possibility of vertices a and b being adjacent. However, the case of a non-adjacent separating pair joining two s -networks has to be excluded since this would imply decomposing a polygon into two smaller ones, which is prohibited. Hence (22). Formula (23) then follows easily, using (20).

The proof of (24) is similar, the difference being that now one of the separated components is also distinguished. Details are left to the reader. \blacksquare

Corollary 10 (Explicit form of the Dissymmetry Theorem) *Let \mathcal{F} be a given species of 3-connected graphs and $\mathcal{B} = \mathcal{B}_{\mathcal{F}}$ be the species of 2-connected graphs all of whose 3-connected components are in \mathcal{F} . Also let $\mathcal{R} = \mathcal{R}_{\mathcal{F}}$ denote the corresponding species of 01-networks. We then have the following identity:*

$$\begin{aligned} \mathcal{B} &= \mathcal{F} \uparrow \mathcal{R} + \mathcal{C} \uparrow (\mathcal{R} - \mathcal{S}) \\ &\quad + K_2 \uparrow (\mathcal{R} - (\mathcal{H} + \mathcal{S})(\mathcal{R} + 1) - E_2(\mathcal{S}) + \mathcal{S}^2), \end{aligned} \quad (25)$$

where \mathcal{R} is characterized by equation (19), $\mathcal{S} = \frac{X\mathcal{R}^2}{1+X\mathcal{R}}$ and $\mathcal{H} = \mathcal{F}_{0,1} \uparrow \mathcal{R}$.

4 Series techniques for the unlabelled enumeration of graphs and networks

Traditionally, two generating series are used for the unlabelled enumeration of structures: the ordinary (tilde) generating function and the cycle index series. These are now reviewed in the context of graphs and networks where the number of vertices and the number of edges are taken into account and a variant of the cycle index series is necessary to deal with the \uparrow -composition, the substitution of networks for edges. This variant, the *edge index series*, is introduced in [25] and called Walsh index series in [8]. Detailed proofs of most of their main properties can be found in [8].

For a species \mathcal{M} of (possibly, rooted) graphs or networks, the ordinary (*tilde*) generating function $\widetilde{\mathcal{M}}(x, y) = \widetilde{\mathcal{M}}(x, y)$ is defined as follows:

$$\widetilde{\mathcal{M}}(x, y) = \sum_{n \geq 0} \tilde{\mu}_n(y) x^n = \sum_{n \geq 0} \sum_{m \geq 0} \tilde{\mu}_{n,m} y^m x^n, \quad (26)$$

where $\tilde{\mu}_{n,m}$ is the number of isomorphism classes of graphs (resp., networks) in \mathcal{M} having n vertices (resp., internal vertices) and m edges.

Let \mathcal{G} be a species of graphs and \mathcal{N} be a species of networks. The three edge index series $W_{\mathcal{G}}(\mathbf{a}; \mathbf{b}; \mathbf{c})$, $W_{\mathcal{N}}^+(\mathbf{a}; \mathbf{b}; \mathbf{c})$ and $W_{\mathcal{N}}^-(\mathbf{a}; \mathbf{b}; \mathbf{c})$ in variables $\mathbf{a} = (a_1, a_2, \dots)$, $\mathbf{b} = (b_1, b_2, \dots)$ and $\mathbf{c} = (c_1, c_2, \dots)$ are defined in what follows.

Let $G = (V(G), E(G))$ be a graph in \mathcal{G} . A permutation σ of $V(G)$, which is an automorphism of the graph G , induces a permutation $\sigma^{(2)}$ of the set $E(G)$ of edges whose cycles are of two possible sorts: if c is a cycle of $\sigma^{(2)}$ of length l , then either $\sigma^l(a) = a$ and $\sigma^l(b) = b$ for each edge $e = ab$ of c (a *cylindrical* edge cycle), or else $\sigma^l(a) = b$ and $\sigma^l(b) = a$ for each edge $e = ab$ of c (a *Möbius* edge cycle). For example, the automorphism $\sigma = (1, 2, 3, 4)(5, 6, 7, 8)$ of the graph in Figure 8 (i) induces the cylindrical edge cycle $(15, 26, 37, 48)$, and the automorphism $\sigma = (1, 2, 3, 4, 5, 6, 7, 8)$ of the graph in Figure 8 (ii) induces the Möbius edge cycle $(15, 26, 37, 48)$.

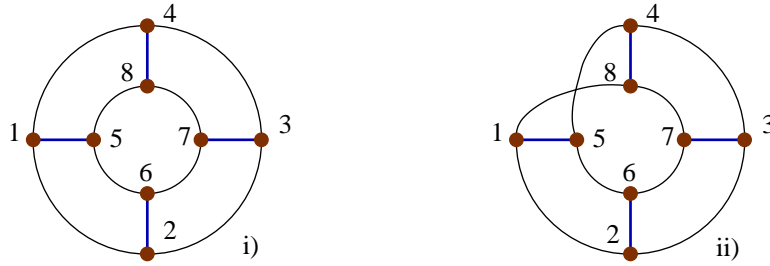


Figure 8: (i) Cylindrical edge cycle, (ii) Möbius edge cycle.

For an automorphism $\sigma \in \text{Aut}(G)$ of G , denote by σ_k the number of cycles of length k of σ , by $\text{cyl}_k(G, \sigma)$ the number of cylindrical edge cycles of length k , and by $\text{möb}_k(G, \sigma)$ the number of Möbius edge cycles of length k induced by σ in G . Given a graph $G \in \mathcal{G}$ and an automorphism σ of G , the *weight* $w(G, \sigma)$ of such a structure is the following cycle index monomial:

$$w(G, \sigma) = a_1^{\sigma_1} a_2^{\sigma_2} \dots b_1^{\text{cyl}_1(G, \sigma)} b_2^{\text{cyl}_2(G, \sigma)} \dots c_1^{\text{möb}_1(G, \sigma)} c_2^{\text{möb}_2(G, \sigma)} \dots \quad (27)$$

The *edge index series* $W_{\mathcal{G}}(\mathbf{a}; \mathbf{b}; \mathbf{c})$ of \mathcal{G} is defined as

$$W_{\mathcal{G}}(\mathbf{a}; \mathbf{b}; \mathbf{c}) = \sum_{G \in \text{Typ}(\mathcal{G})} \frac{1}{|\text{Aut}(G)|} \sum_{\sigma \in \text{Aut}(G)} w(G, \sigma), \quad (28)$$

where the notation $G \in \text{Typ}(\mathcal{G})$ means that the summation should be taken over a set of representatives G of the isomorphism classes of graphs in \mathcal{G} .

Examples. 1. The edge index series of the species K_2 is given by

$$W_{K_2} = \frac{1}{2}(a_1^2 b_1 + a_2 c_1). \quad (29)$$

2. The edge index series of the species \mathcal{C}_n of (unoriented) cycles of length n is a refinement of the usual cycle index polynomial for \mathcal{C}_n . It is given by

$$W_{\mathcal{C}_n}(\mathbf{a}; \mathbf{b}; \mathbf{c}) = \frac{1}{2n} \sum_{d|n} \phi(d) a_d^{\frac{n}{d}} b_d^{\frac{n}{d}} + \frac{1}{2} \begin{cases} a_1 a_2^{\frac{n-1}{2}} b_2^{\frac{n-1}{2}} c_1, & n \text{ odd} \\ \frac{1}{2}(a_2^{\frac{n}{2}} b_2^{\frac{n-2}{2}} c_1^2 + a_1^2 a_2^{\frac{n-2}{2}} b_2^{\frac{n}{2}}), & n \text{ even} \end{cases} \quad (30)$$

where ϕ is the Euler ϕ -function. See [25] and [8] which contains a typo in formula (43). By summing over $n \geq 3$, we obtain the edge index series of \mathcal{C} . The result is

$$W_{\mathcal{C}} = \frac{1}{2} \sum_{d \geq 1} \frac{\phi(d)}{d} \log \frac{1}{1 - a_d b_d} - \frac{1}{2} a_1 b_1 - \frac{1}{4} a_1^2 b_1^2 - \frac{1}{4} a_2 b_2 + \frac{1}{4} (2a_1 c_1 + a_2 c_1^2 + a_1^2 b_2) \frac{a_2 b_2}{1 - a_2 b_2}. \quad (31)$$

Note that any isomorphism of networks $\varphi : N \xrightarrow{\sim} N'$ is assumed to be pole-preserving, i.e. $\varphi(0) = 0$ and $\varphi(1) = 1$. In particular, any automorphism of a network N should be pole-preserving. It will be necessary to consider the subclass \mathcal{N}_τ of \mathcal{N} consisting of τ -symmetric networks, i.e.

$$\mathcal{N}_\tau = \{N \in \mathcal{N} \mid \tau \cdot N \simeq N\}. \quad (32)$$

Let U be the underlying set of a network N , and suppose that σ is in $S[U]$, i.e. σ is a permutation of U . We can extend σ to permutations on $U \cup \{0, 1\}$, $\sigma^+ = (0)(1)\sigma$ and $\sigma^- = (0, 1)\sigma$. In other words, σ^+ preserves the poles and σ^- exchanges them. For any network $N \in \mathcal{N}$, denote by \hat{N} the corresponding graph on $U \cup \{0, 1\}$. Then we introduce notation

$$\text{Aut}^+(N) = \{\sigma \in S[U] \mid \sigma^+ \in \text{Aut}(\hat{N})\} \quad (33)$$

and

$$\text{Aut}^-(N) = \{\sigma \in S[U] \mid \sigma^- \in \text{Aut}(\hat{N})\}. \quad (34)$$

In other words, a σ^+ in (33) is a pole-preserving graph automorphism, i.e. a network automorphism, while a σ^- in (34) is a pole-reversing *graph* automorphism. Notice that $\text{Aut}^+(N) = \text{Aut}(N)$ and, if $\text{Aut}^-(N)$ is not empty, then $|\text{Aut}^-(N)| = |\text{Aut}^+(N)|$. This can be seen by using the composition of automorphisms. For $N \in \mathcal{N}$ and $\sigma \in \text{Aut}^+(N)$, we assign the weight

$$w(N, \sigma) = \frac{w(\hat{N}, \sigma^+)}{a_1^2}, \quad (35)$$

where $w(\hat{N}, \sigma^+)$ is defined by (27), and for $N \in \mathcal{N}$ and $\sigma \in \text{Aut}^-(N)$, we set

$$w(N, \sigma) = \frac{w(\hat{N}, \sigma^-)}{a_2}. \quad (36)$$

In other words, only the internal vertex cycles are accounted for. Then, for a species \mathcal{N} of networks, the two *edge index series* are defined by

$$W_{\mathcal{N}}^+(\mathbf{a}; \mathbf{b}; \mathbf{c}) = \sum_{N \in \text{Typ}(\mathcal{N})} \frac{1}{|\text{Aut}^+(N)|} \sum_{\sigma \in \text{Aut}^+(N)} w(N, \sigma), \quad (37)$$

$$W_{\mathcal{N}}^-(\mathbf{a}; \mathbf{b}; \mathbf{c}) = \sum_{N \in \text{Typ}(\mathcal{N}_\tau)} \frac{1}{|\text{Aut}^-(N)|} \sum_{\sigma \in \text{Aut}^-(N)} w(N, \sigma). \quad (38)$$

As the next proposition shows, the edge index series contain all the enumerative (labelled and unlabelled) information.

Proposition 11 ([8, 25]) *Let \mathcal{G} be a species of graphs and \mathcal{N} be a species of networks. Then the following series identities hold:*

$$\mathcal{G}(x, y) = W_{\mathcal{G}}(x, 0, 0, \dots; y, y^2, y^3, \dots; y, y^2, y^3, \dots), \quad (39)$$

$$\tilde{\mathcal{G}}(x, y) = W_{\mathcal{G}}(x, x^2, x^3, \dots; y, y^2, y^3, \dots; y, y^2, y^3, \dots), \quad (40)$$

$$\mathcal{N}(x, y) = W_{\mathcal{N}}^+(x, 0, 0, \dots; y, y^2, y^3, \dots; y, y^2, y^3, \dots), \quad (41)$$

$$\tilde{\mathcal{N}}(x, y) = W_{\mathcal{N}}^+(x, x^2, x^3, \dots; y, y^2, y^3, \dots; y, y^2, y^3, \dots), \quad (42)$$

$$\mathcal{N}_{\tau}(x, y) = W_{\mathcal{N}}^-(x, 0, 0, \dots; y, y^2, y^3, \dots; y, y^2, y^3, \dots), \quad (43)$$

$$\tilde{\mathcal{N}}_{\tau}(x, y) = W_{\mathcal{N}}^-(x, x^2, x^3, \dots; y, y^2, y^3, \dots; y, y^2, y^3, \dots). \quad (44)$$

■

Another description of the edge index series is very useful for understanding them and for establishing their properties. It consists of expressions which involve exponential generating functions of labelled enumeration. These are recalled from Section 6 of [8].

Following an idea of Joyal [11], we introduce the auxiliary weighted species $\mathcal{G}^{\text{aut}} = \mathcal{G}_w^{\text{aut}}$. For any finite set U (of vertices), $\mathcal{G}^{\text{aut}}[U]$ is defined as the set of graphs in $\mathcal{G}[U]$ equipped with an automorphism σ , i.e.

$$\mathcal{G}^{\text{aut}}[U] = \{(G, \sigma) \mid G \in \mathcal{G}[U], \sigma \in S[U] : \sigma \cdot G = G\},$$

where $S[U]$ is the set of all permutations of U . The relabelling rule of \mathcal{G}^{aut} -structures along a bijection $\beta : U \xrightarrow{\sim} U'$ is defined as follows:

$$\beta \cdot (G, \sigma) = (\beta \cdot G, \beta \circ \sigma \circ \beta^{-1}),$$

where $\beta \cdot G$ is the graph obtained from G by relabelling along β and the composition \circ is taken from right to left. It is easy to verify that $\mathcal{G}_w^{\text{aut}}$ is a well-defined weighted species, where the weight function $w(G, \sigma)$ is the cycle index monomial defined by (27). Recall that $|\mathcal{G}^{\text{aut}}[n]|_w$ denotes the total weight of $\mathcal{G}_w^{\text{aut}}$ -structures over the vertex set $[n] := \{1, 2, \dots, n\}$, i.e.

$$|\mathcal{G}^{\text{aut}}[n]|_w = \sum_{(G, \sigma) \in \mathcal{G}_w^{\text{aut}}[n]} w(G, \sigma).$$

Proposition 12 ([8]) *Using the exponential generating function of labelled $\mathcal{G}_w^{\text{aut}}$ -structures, we have*

$$W_{\mathcal{G}}(\mathbf{a}; \mathbf{b}; \mathbf{c}) = \sum_{n \geq 0} \frac{1}{n!} |\mathcal{G}^{\text{aut}}[n]|_w = \mathcal{G}_w^{\text{aut}}(x) |_{x=1}. \quad (45)$$

Proof. The proof follows from the fact that the number of distinct graphs on $[n]$ obtained by relabelling a given graph G with n vertices is given by $\frac{n!}{|\text{Aut}(G)|}$. ■

A similar approach can be used for the edge index series $W_{\mathcal{N}}^+$ and $W_{\mathcal{N}}^-$ of a given species of 2-pole networks \mathcal{N} . We introduce the sets

$$\mathcal{N}^+[U] = \{(N, \sigma) \mid N \in \mathcal{N}[U], \sigma \in \text{Aut}^+(N)\}$$

and

$$\mathcal{N}^-[U] = \{(N, \sigma) \mid N \in \mathcal{N}[U], \sigma \in \text{Aut}^-(N)\},$$

where $\text{Aut}^+(N)$ and $\text{Aut}^-(N)$ are defined by (33) and (34), respectively. Then, using the weight functions given by (35) and (36), \mathcal{N}_w^+ and \mathcal{N}_w^- are weighted species whose labelled enumeration yields by specialization the series $W_{\mathcal{N}}^+$ and $W_{\mathcal{N}}^-$, respectively.

Proposition 13 ([8]) *For a species of networks \mathcal{N} , the edge index series $W_{\mathcal{N}}^+$ and $W_{\mathcal{N}}^-$ can be expressed by the formulas*

$$W_{\mathcal{N}}^+(\mathbf{a}; \mathbf{b}; \mathbf{c}) = \mathcal{N}_w^+(x)|_{x=1}, \quad W_{\mathcal{N}}^-(\mathbf{a}; \mathbf{b}; \mathbf{c}) = \mathcal{N}_w^-(x)|_{x=1}. \quad (46)$$

■

In order to describe the edge index series of an \uparrow -composition, we introduce the following familiar plethystic notation. For any series of edge index type $f(\mathbf{a}; \mathbf{b}; \mathbf{c})$ and any integer $k \geq 1$, we set

$$f_k = f_k(\mathbf{a}; \mathbf{b}; \mathbf{c}) = f(a_k, a_{2k}, a_{3k}, \dots; b_k, b_{2k}, b_{3k}, \dots; c_k, c_{2k}, c_{3k}, \dots). \quad (47)$$

Moreover, for any series $\ell = \ell(\mathbf{a}; \mathbf{b}; \mathbf{c})$, $f = f(\mathbf{a}; \mathbf{b}; \mathbf{c})$, $g = g(\mathbf{a}; \mathbf{b}; \mathbf{c})$, and $h = h(\mathbf{a}; \mathbf{b}; \mathbf{c})$, we set

$$\ell[f; g; h](\mathbf{a}; \mathbf{b}; \mathbf{c}) = \ell(f_1, f_2, f_3, \dots; g_1, g_2, g_3, \dots; h_1, h_2, h_3, \dots), \quad (48)$$

and for series $\alpha = \alpha(x, y)$, $\beta = \beta(x, y)$, and $\gamma = \gamma(x, y)$, we also set

$$\begin{aligned} \ell[\alpha; \beta; \gamma](x, y) &= \ell(\alpha(x, y), \alpha(x^2, y^2), \alpha(x^3, y^3), \dots; \beta(x, y), \beta(x^2, y^2), \beta(x^3, y^3), \dots; \\ &\quad \gamma(x, y), \gamma(x^2, y^2), \gamma(x^3, y^3), \dots). \end{aligned} \quad (49)$$

Theorem 14 ([8, 25]) *Let \mathcal{G} be a species of graphs and \mathcal{N} be a symmetric species of networks. Then the edge index series and the tilde series of the species $\mathcal{G} \uparrow \mathcal{N}$ are given by*

$$\begin{aligned} W_{\mathcal{G} \uparrow \mathcal{N}}(\mathbf{a}; \mathbf{b}; \mathbf{c}) &= W_{\mathcal{G}}(a_1, a_2, \dots; W_{\mathcal{N}}^+, W_{\mathcal{N},2}^+, \dots; W_{\mathcal{N}}^-, W_{\mathcal{N},2}^-, \dots) \\ &= W_{\mathcal{G}}[a_1; W_{\mathcal{N}}^+; W_{\mathcal{N}}^-] \end{aligned} \quad (50)$$

and

$$\begin{aligned} (\mathcal{G} \uparrow \mathcal{N})^\sim(x, y) &= W_{\mathcal{G}}(x, x^2, \dots; \tilde{\mathcal{N}}(x, y), \tilde{\mathcal{N}}(x^2, y^2), \dots; \tilde{\mathcal{N}}_\tau(x, y), \tilde{\mathcal{N}}_\tau(x^2, y^2), \dots) \\ &= W_{\mathcal{G}}[x; \tilde{\mathcal{N}}(x, y); \tilde{\mathcal{N}}_\tau(x, y)]. \end{aligned} \quad (51)$$

Proof. See [8]. ■

Similarly, for a composition of networks $\mathcal{M} \uparrow \mathcal{N}$, we have the following.

Theorem 15 ([25]) *Let \mathcal{M} be a species of networks and \mathcal{N} be a symmetric species of networks. Then the edge index series and the tilde series of the species $\mathcal{M} \uparrow \mathcal{N}$ are given by*

$$W_{\mathcal{M} \uparrow \mathcal{N}}^+(\mathbf{a}; \mathbf{b}; \mathbf{c}) = W_{\mathcal{M}}^+[a_1; W_{\mathcal{N}}^+; W_{\mathcal{N}}^-], \quad (52)$$

$$W_{\mathcal{M} \uparrow \mathcal{N}}^-(\mathbf{a}; \mathbf{b}; \mathbf{c}) = W_{\mathcal{M}}^-[a_1; W_{\mathcal{N}}^+; W_{\mathcal{N}}^-], \quad (53)$$

and

$$(\mathcal{M} \uparrow \mathcal{N})^\sim(x, y) = W_{\mathcal{M}}^+[x; \tilde{\mathcal{N}}(x, y); \tilde{\mathcal{N}}_\tau(x, y)], \quad (54)$$

$$(\mathcal{M} \uparrow \mathcal{N})^\sim_\tau(x, y) = W_{\mathcal{M}}^-[x; \tilde{\mathcal{N}}(x, y); \tilde{\mathcal{N}}_\tau(x, y)]. \quad (55)$$

Proof. The proof is similar to that of Theorem 14 which is given in [8]. For (52) and (53), one uses the fact that the poles of an $\mathcal{M} \uparrow \mathcal{N}$ -structure are preserved (resp., exchanged) if and only if the poles of its core are preserved (resp., exchanged). Notice that (54) and (55) are consequences of (52) and (53) by virtue of Proposition 11. ■

Proposition 16 ([8, 25]) *Let \mathcal{B} be a species of 2-connected graphs and $K_2 \in \mathcal{B}$. Then the edge index series of the associated species of networks $\mathcal{B}_{0,1}$ and $\mathcal{N}_{\mathcal{B}} = (1 + y)\mathcal{B}_{0,1} - \mathbb{1}$ are given by*

$$W_{\mathcal{B}_{0,1}}^+(\mathbf{a}; \mathbf{b}; \mathbf{c}) = \frac{2}{a_1^2} \frac{\partial}{\partial b_1} W_{\mathcal{B}}(\mathbf{a}; \mathbf{b}; \mathbf{c}), \quad (56)$$

$$W_{\mathcal{B}_{0,1}}^-(\mathbf{a}; \mathbf{b}; \mathbf{c}) = \frac{2}{a_2} \frac{\partial}{\partial c_1} W_{\mathcal{B}}(\mathbf{a}; \mathbf{b}; \mathbf{c}), \quad (57)$$

$$W_{\mathcal{N}_{\mathcal{B}}}^+(\mathbf{a}; \mathbf{b}; \mathbf{c}) = (1 + b_1)W_{\mathcal{B}_{0,1}}^+(\mathbf{a}; \mathbf{b}; \mathbf{c}) - 1, \quad (58)$$

$$W_{\mathcal{N}_{\mathcal{B}}}^-(\mathbf{a}; \mathbf{b}; \mathbf{c}) = (1 + c_1)W_{\mathcal{B}_{0,1}}^-(\mathbf{a}; \mathbf{b}; \mathbf{c}) - 1. \quad (59)$$

Notice that, given a species \mathcal{N} of networks with non-adjacent poles, for the operator $\mathcal{N} \mapsto y\mathcal{N}$, which consists in adding the edge 01 to all networks in \mathcal{N} , we have

$$W_{y\mathcal{N}}^+ = b_1 W_{\mathcal{N}}^+ \quad \text{and} \quad W_{y\mathcal{N}}^- = c_1 W_{\mathcal{N}}^-. \quad (60)$$

For the series composition of networks, we have the following edge index series identities.

Theorem 17 *Let \mathcal{M} and \mathcal{N} be species of non-trivial networks. Then we have*

$$W_{\mathcal{M} \cdot_s \mathcal{N}}^+(\mathbf{a}; \mathbf{b}; \mathbf{c}) = a_1 W_{\mathcal{M}}^+(\mathbf{a}; \mathbf{b}; \mathbf{c}) W_{\mathcal{N}}^+(\mathbf{a}; \mathbf{b}; \mathbf{c}), \quad (61)$$

$$W_{\mathcal{M} \cdot_s \mathcal{M}}^-(\mathbf{a}; \mathbf{b}; \mathbf{c}) = a_1 W_{\mathcal{M},2}^+(\mathbf{a}; \mathbf{b}; \mathbf{c}), \quad (62)$$

$$W_{\mathcal{M} \cdot_s \mathcal{N} \cdot_s \mathcal{M}}^-(\mathbf{a}; \mathbf{b}; \mathbf{c}) = a_2 W_{\mathcal{M},2}^+(\mathbf{a}; \mathbf{b}; \mathbf{c}) W_{\mathcal{N}}^-(\mathbf{a}; \mathbf{b}; \mathbf{c}). \quad (63)$$

Proof. We can use the representations (46) for the edge index series W^+ and W^- which interpret these series as exponential generating functions of labelled structures. Thus, in the first case, we are interested in the exponential generating function $(\mathcal{M} \cdot_s \mathcal{N})_w^+(x)$. Now, an $\mathcal{M} \cdot_s \mathcal{N}$ -structure is a pair $(M \cdot_s N, \sigma)$, where $M \cdot_s N$ is a series composition with $M \in \mathcal{M}$ and $N \in \mathcal{N}$ and σ is a pole-preserving automorphism of $M \cdot_s N$. It is clear that

the connecting vertex of the series composition is left fixed by σ , and σ can be restricted to pole-preserving automorphisms σ_M and σ_N of M and N , respectively. Moreover, for the weight w defined by (35), we have

$$w(M \cdot_s N, \sigma) = a_1 w(M, \sigma_M) w(N, \sigma_N).$$

Hence the generating functions satisfy

$$(\mathcal{M} \cdot_s \mathcal{N})_w^+(x) = x a_1 M_w^+(x) N_w^+(x),$$

and (61) follows.

In order to prove (62), one should enumerate structures of the form $(M \cdot_s M', \sigma)$, where $M \cdot_s M'$ is a τ -symmetric series-composition network with M and M' in \mathcal{M} , and σ is a pole-reversing graph automorphism. In this case σ will leave the connecting vertex c fixed and will induce two network isomorphisms

$$\varphi = \sigma|_M : M \xrightarrow{\sim} \tau M' \quad \text{and} \quad \rho = \sigma^2|_M : M \xrightarrow{\sim} M. \quad (64)$$

See Figure 9 where the isomorphism φ is represented as $x \mapsto x'$, for $x = a, d, e, f$. Conversely, the data of ρ and φ determines σ since $\sigma = \varphi \cup (c) \cup \rho \circ \varphi^{-1}$. Moreover, all the vertex- and edge-cycles of ρ have their lengths doubled in σ . For example, taking $\rho = (a)(d, e, f)$ in Figure 9, we find that $\sigma = (a, a')(d, d', e, e', f, f')$. It follows that

$$(\mathcal{M} \cdot_s \mathcal{M})_w^-(x) = x a_1 M_{(w)_2}^+(x^2),$$

where we set $(w)_2(G, \sigma) = w(G, \sigma)_2$, corresponding to the plethystic notation (47).

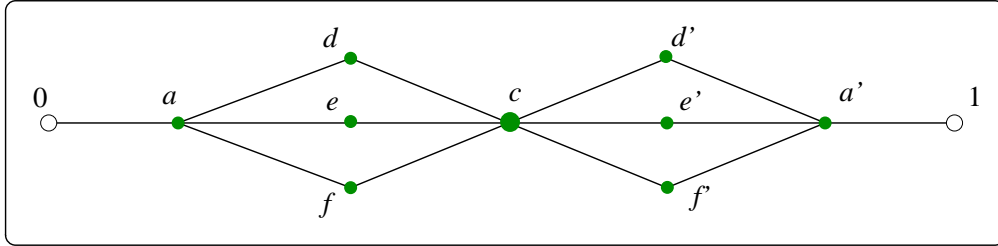


Figure 9: τ -symmetric series composition of networks.

In the case of (63), the reasoning is similar. Here a pole-reversing automorphism σ of a series composition $M \cdot_s N \cdot_s M'$ will exchange M and M' and, furthermore, induce a pole-reversing automorphism of N and interchange the two connecting vertices. Details are left to the reader. ■

Proposition 18 *Let $\mathcal{R} = \mathcal{R}_{\mathcal{F}}$ be the class of non-trivial networks all of whose 3-connected components are in a given species \mathcal{F} , and denote by \mathcal{S} the class of s -networks in \mathcal{R} . Then*

$$W_{\mathcal{S}}^+(\mathbf{a}; \mathbf{b}; \mathbf{c}) = \frac{a_1 (W_{\mathcal{R}}^+)^2}{1 + a_1 W_{\mathcal{R}}^+} \quad (65)$$

and

$$W_{\mathcal{S}}^-(\mathbf{a}; \mathbf{b}; \mathbf{c}) = \frac{(a_1 + a_2 W_{\mathcal{R}}^-) W_{\mathcal{R},2}^+}{1 + a_2 W_{\mathcal{R},2}^+}. \quad (66)$$

Proof. From Proposition 3, we have

$$\mathcal{S} = (\mathcal{R} - \mathcal{S}) \cdot_s \mathcal{R} = \mathcal{R} \cdot_s \mathcal{R} - \mathcal{S} \cdot_s \mathcal{R}, \quad (67)$$

and, by Theorem 17,

$$W_{\mathcal{S}}^+ = a_1(W_{\mathcal{R}}^+)^2 - a_1 W_{\mathcal{S}}^+ W_{\mathcal{R}}^+. \quad (68)$$

Solving for $W_{\mathcal{S}}^+$, we obtain (65). However, formula (67) can not be used for computing the edge index series $W_{\mathcal{S}}^-$ since the decomposition is not preserved by a pole-reversing automorphism. One should rather use the more symmetric canonical decomposition (11) and then apply (62) and (63). Regrouping the even and the odd \cdot_s -powers yields

$$W_{\mathcal{S}}^- = \frac{a_1(W_{\mathcal{R},2}^+ - W_{\mathcal{S},2}^+)}{1 - a_2(W_{\mathcal{R},2}^+ - W_{\mathcal{S},2}^+)} + \frac{a_2(W_{\mathcal{R},2}^+ - W_{\mathcal{S},2}^+)(W_{\mathcal{R}}^- - W_{\mathcal{S}}^-)}{1 - a_2(W_{\mathcal{R},2}^+ - W_{\mathcal{S},2}^+)}, \quad (69)$$

and, after simplification,

$$W_{\mathcal{S}}^- = (W_{\mathcal{R},2}^+ - W_{\mathcal{S},2}^+)(a_1 + a_2 W_{\mathcal{R}}^-). \quad (70)$$

Formula (65) can then be used, and the result follows. \blacksquare

Theorem 19 *Let \mathcal{N} be a species of non-trivial networks having non-adjacent poles. Then the edge index series of the species of parallel compositions $E(\mathcal{N})$ are given by*

$$W_{E(\mathcal{N})}^+(\mathbf{a}; \mathbf{b}; \mathbf{c}) = \exp \left(\sum_{m=1}^{\infty} \frac{W_{\mathcal{N},m}^+}{m} \right) \quad (71)$$

and

$$W_{E(\mathcal{N})}^-(\mathbf{a}; \mathbf{b}; \mathbf{c}) = \exp \left(\sum_{m \text{ even}} \frac{W_{\mathcal{N},m}^+}{m} + \sum_{m \text{ odd}} \frac{W_{\mathcal{N},m}^-}{m} \right). \quad (72)$$

We also have

$$W_{E_2(\mathcal{N})}^+ = \frac{1}{2}((W_{\mathcal{N}}^+)^2 + W_{\mathcal{N},2}^+) \quad \text{and} \quad W_{E_2(\mathcal{N})}^- = \frac{1}{2}((W_{\mathcal{N}}^-)^2 + W_{\mathcal{N},2}^+). \quad (73)$$

Proof. We use again the representations (46) for the edge index series W^+ and W^- . In the first case, we are interested in the exponential generating function $E(\mathcal{N})_w^+(x)$. An $E(\mathcal{N})^+$ -structure consists of a parallel composition of networks in \mathcal{N} together with a network automorphism σ . This σ induces a permutation σ_0 on the set of individual networks involved in the parallel composition. Decomposing σ_0 into (oriented) cycles yields a natural notion of *connected* $E(\mathcal{N})^+$ -structure, namely, when σ_0 is a circular permutation leading to an oriented cycle of network isomorphisms. These are known as *cylindrical m -wreaths of networks* (see [11, 8]),

$$c_m : N_1 \xrightarrow{\varphi_1} N_2 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_{m-2}} N_{m-1} \xrightarrow{\varphi_{m-1}} N_m \xrightarrow{\varphi_m} N_1, \quad (74)$$

where $m \geq 1$. Denote by $\mathcal{K}_m(\mathcal{N})$ the species of cylindrical m -wreaths of networks and by $\mathcal{K}_m^\bullet(\mathcal{N})$ the species of *rooted* cylindrical m -wreaths of networks, where one network

is distinguished from the others. In fact, description (74) includes a rooting at N_1 . In the unrooted case, all the possible rootings are considered equivalent. It follows that any $E(\mathcal{N})^+$ -structure can be seen as an assembly of (unrooted) cylindrical wreaths of networks, and we have a weighted species isomorphism

$$E(\mathcal{N})_w^+ = E\left(\sum_{m \geq 1} \mathcal{K}_m(\mathcal{N})_w\right). \quad (75)$$

Given a rooted cylindrical m -wreath of networks c_m in $\mathcal{K}_m^\bullet(\mathcal{N})$ of the form (74), the composite $\varphi_0 = \varphi_m \circ \varphi_{m-1} \circ \dots \circ \varphi_2 \circ \varphi_1$ is an automorphism of N_1 , and we obtain a \mathcal{N}^+ -structure (N_1, φ_0) . Moreover, the sequence of network isomorphisms $(\varphi_1, \dots, \varphi_{m-1})$ can be encoded in a set of lists of length m , (u_1, u_2, \dots, u_m) , where u_1 runs over the underlying set of N_1 and $u_{i+1} = \varphi_i(u_i)$, $i = 1 \dots m-1$, and we can consider the \mathcal{N}^+ -structure (N_1, φ_0) to “live” on this set of lists. In other words, what we have obtained is an $\mathcal{N}^+(X^m)$ -structure. Since the isomorphism φ_m can be recovered from φ_0 and the other isomorphisms φ_i , this correspondence is bijective. Moreover, the weight of the connected $E(\mathcal{N})^+$ -structure c_m is given by $(w)_m(N_1, \varphi_0)$ since all the cycle lengths of φ_0 are multiplied by m in φ_0 . Hence we have an isomorphism of weighted species (see also Proposition 14 of [8])

$$\mathcal{K}_m^\bullet(\mathcal{N})_w = \mathcal{N}_{(w)_m}^+(X^m) \quad (76)$$

and the exponential generating function equality

$$\mathcal{K}_m(\mathcal{N})_w(x) = \frac{1}{m} \mathcal{K}_m^\bullet(\mathcal{N})_w(x) = \frac{1}{m} \mathcal{N}_{(w)_m}^+(x^m). \quad (77)$$

Using (75) and the classical exponential formula, we find that

$$E(\mathcal{N})_w^+(x) = \exp\left(\sum_{m \geq 1} \frac{1}{m} \mathcal{N}_{(w)_m}^+(x^m)\right), \quad (78)$$

and (71) follows.

For (72), one should compute the exponential generating function $E(\mathcal{N})_w^-(x)$. An $E(\mathcal{N})^-$ -structure consists of a parallel composition of networks in \mathcal{N} together with a pole-reversing automorphism σ . Here two kinds of connected components can occur. The first kind arises from a cylindrical m -wreath of networks such as (74), with m even, which is reinterpreted as a sequence of pole-reversing network isomorphisms

$$N_1 \xrightarrow{\varphi_1} \tau N_2 \xrightarrow{\varphi_2} N_3 \xrightarrow{\varphi_3} \dots \xrightarrow{\varphi_{m-2}} N_{m-1} \xrightarrow{\varphi_{m-1}} \tau N_m \xrightarrow{\varphi_m} N_1. \quad (79)$$

Here also the composite $\varphi_0 = \varphi_m \circ \varphi_{m-1} \circ \dots \circ \varphi_2 \circ \varphi_1$ is an automorphism of N_1 and this accounts for the first term on the right-hand side of (72).

The second kind of connected component corresponds to a *Möbius m -wreath of networks*, with m odd, which is defined as a sequence of network isomorphisms $N_1 \xrightarrow{\varphi_1} N_2 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_{m-2}} N_{m-1} \xrightarrow{\varphi_{m-1}} N_m$ followed by a pole-reversing isomorphism $\varphi_m : N_m \longrightarrow \tau N_1$, which can be reinterpreted as a sequence of pole-reversing isomorphisms

$$N_1 \xrightarrow{\varphi_1} \tau N_2 \xrightarrow{\varphi_2} N_3 \xrightarrow{\varphi_3} \dots \xrightarrow{\varphi_{m-2}} \tau N_{m-1} \xrightarrow{\varphi_{m-1}} N_m \xrightarrow{\varphi_m} \tau N_1. \quad (80)$$

Notice that the composite $\varphi_0 = \varphi_m \circ \varphi_{m-1} \circ \dots \circ \varphi_2 \circ \varphi_1$ is a pole-reversing automorphism of N_1 , and this accounts for the second term on the right-hand side of (72). See also [8].

The proof of (73) relies on the fact that

$$E_2(\mathcal{N})_w^+ = E_2(\mathcal{N}_w^+) + \mathcal{K}_2(\mathcal{N})_w \quad \text{and} \quad E_2(\mathcal{N})_w^- = E_2(\mathcal{N}_w^-) + \mathcal{K}_2(\mathcal{N})_w. \quad (81)$$

Details are left to the reader. ■

Remark. A proof of (71) and (72) obtained by expressing a parallel composition as an \uparrow -composition whose core is a “network” with parallel edges and no internal vertices appears in [25].

5 Enumerative applications

Again, let \mathcal{F} be a given class of 3-connected graphs and denote by $\mathcal{B} = \mathcal{B}_{\mathcal{F}}$ the class of 2-connected graphs all of whose 3-connected components are in \mathcal{F} . Also, denote by $\mathcal{R} = \mathcal{R}_{\mathcal{F}}$ the class of networks all of whose 3-connected components are in \mathcal{F} . Given \mathcal{F} , the species \mathcal{R} can be determined recursively, as well as its associated series, from the fundamental relations of Theorem 8. Using the dissymmetry theorem, the species \mathcal{B} and its series can also be determined.

Formulas of the previous sections can be applied to obtain both labelled and unlabelled enumeration of species of 2- or 3-connected graphs. The labelled enumeration is usually simpler because it is not necessary to use the Dissymmetry Theorem in order to unroot the structures, and the composition formulas are simpler for the exponential generating functions. For the unlabelled enumeration, the formulas are more delicate and are reviewed below. Some standard applications and new ones are presented as well.

5.0.1 Labelled enumeration

For the network species $\mathcal{R} = \mathcal{R}_{\mathcal{F}}$, we deduce the following functional equation for the exponential generating function:

$$\mathcal{R}(x, y) = (1 + y) \exp \left(\mathcal{F}_{0,1}(x, \mathcal{R}(x, y)) + \frac{x\mathcal{R}^2(x, y)}{1 + x\mathcal{R}(x, y)} \right) - 1. \quad (82)$$

Setting $g(x, y) = \mathcal{F}_{0,1}(x, y) + \frac{xy^2}{1+xy}$, and $\zeta(x, y) = \mathcal{R}(x, y)^{\langle -1 \rangle_y}$, the compositional inverse of $\mathcal{R}(x, y)$ with respect to y , we have, from (82),

$$1 + \mathcal{R}(x, y) = (1 + y) \exp(g(x, \mathcal{R}(x, y))) \quad (83)$$

and

$$1 + \zeta(x, y) = (1 + y) \exp(-g(x, y)). \quad (84)$$

Notice that $\zeta(x, y)$ is of the form

$$\zeta(x, y) = y \left(1 + (1 + y) \frac{\exp(-g(x, y)) - 1}{y} \right), \quad (85)$$

so that Lagrange inversion can be used to find $\mathcal{R}(x, y)$, knowing $\mathcal{F}_{0,1}(x, y)$.

Conversely, taking logarithms in (84) yields $\mathcal{F}_{0,1}(x, y)$ in terms of $\mathcal{R}(x, y)$:

$$\mathcal{F}_{0,1}(x, y) = \log \frac{1+y}{1+\zeta(x, y)} - \frac{xy^2}{1+xy}. \quad (86)$$

Finally, note that

$$\mathcal{F}(x, y) = \frac{x^2}{2} \int \mathcal{F}_{0,1}(x, y) dy \quad \text{and} \quad \mathcal{B}(x, y) = \frac{x^2}{2} \int \frac{1+\mathcal{R}(x, y)}{1+y} dy. \quad (87)$$

This is essentially the approach used in [24] for the enumeration of labelled 3-connected graphs starting with 1- and 2-connected graphs, and in [2], where labelled 2-connected planar graphs are enumerated starting with 3-connected planar graphs.

In general, the dissymmetry theorem is most useful for unlabelled enumeration (see for example [5, 13]). However, in the present case, taking the exponential generating function of (25) gives a very interesting formula, first found by Giménez, Noy and Rué (see formula (2) of [9]) and essential in their asymptotic analysis of graph classes with given 3-connected components, namely,

$$\begin{aligned} \mathcal{B}(x, y) &= \mathcal{F}(x, \mathcal{R}(x, y)) + \frac{x^2}{2} \left(\mathcal{R}(x, y) + \frac{1}{2} \mathcal{R}^2(x, y) \right) \\ &+ \frac{x^2}{2} \left((1 + \mathcal{R}(x, y)) \log \frac{1+y}{1+\mathcal{R}(x, y)} \right) - \frac{1}{2} x \mathcal{R}(x, y) + \frac{1}{2} \log(1 + x \mathcal{R}(x, y)). \end{aligned} \quad (88)$$

In order to see this, the following observations are useful. First, the exponential generating function of the species \mathcal{C} of non-oriented cycles of length at least 3 is given by

$$\mathcal{C}(x, y) = \frac{1}{2} \left(\log \frac{1}{1-xy} - xy - \frac{x^2 y^2}{2} \right). \quad (89)$$

Also, it follows from (20) that

$$\mathcal{H}(x, y) + \mathcal{S}(x, y) = \log \frac{1+\mathcal{R}(x, y)}{1+y}. \quad (90)$$

Finally, note that

$$\mathcal{R}(x, y) - \mathcal{S}(x, y) = \mathcal{R}(x, y) - \frac{x \mathcal{R}^2(x, y)}{1+x \mathcal{R}(x, y)} = \frac{\mathcal{R}(x, y)}{1+x \mathcal{R}(x, y)}, \quad (91)$$

and that

$$\frac{1}{1-x(\mathcal{R}-\mathcal{S})(x, y)} = 1+x \mathcal{R}(x, y). \quad (92)$$

Details are left to the reader.

5.0.2 Unlabelled enumeration

We introduce the following abbreviations for the edge index series of the species \mathcal{R} , \mathcal{S} of s -networks, and $\mathcal{H} = \mathcal{F}_{0,1} \uparrow \mathcal{R}$ of h -networks:

$$\rho^+(\mathbf{a}; \mathbf{b}; \mathbf{c}) = W_{\mathcal{R}}^+(\mathbf{a}; \mathbf{b}; \mathbf{c}), \quad \rho^-(\mathbf{a}; \mathbf{b}; \mathbf{c}) = W_{\mathcal{R}}^-(\mathbf{a}; \mathbf{b}; \mathbf{c}), \quad (93)$$

$$\sigma^+(\mathbf{a}; \mathbf{b}; \mathbf{c}) = W_{\mathcal{S}}^+(\mathbf{a}; \mathbf{b}; \mathbf{c}), \quad \sigma^-(\mathbf{a}; \mathbf{b}; \mathbf{c}) = W_{\mathcal{S}}^-(\mathbf{a}; \mathbf{b}; \mathbf{c}), \quad (94)$$

and

$$\eta^+(\mathbf{a}; \mathbf{b}; \mathbf{c}) = W_H^+(\mathbf{a}; \mathbf{b}; \mathbf{c}), \quad \eta^-(\mathbf{a}; \mathbf{b}; \mathbf{c}) = W_H^-(\mathbf{a}; \mathbf{b}; \mathbf{c}). \quad (95)$$

It follows from equations (15), (16), (19), and the properties of the edge index series that

$$\rho^+ = (1 + b_1) \exp\left(\sum_{i=1}^{\infty} \frac{\eta_i^+ + \sigma_i^+}{i}\right) - 1, \quad (96)$$

$$\rho^- = (1 + c_1) \exp\left(\sum_{i \text{ even}} \frac{\eta_i^+ + \sigma_i^+}{i} + \sum_{i \text{ odd}} \frac{\eta_i^- + \sigma_i^-}{i}\right) - 1, \quad (97)$$

$$\eta^+ = W_{\mathcal{F}_{0,1}}^+[a_1; \rho^+; \rho^-], \quad \eta^- = W_{\mathcal{F}_{0,1}}^-[a_1; \rho^+; \rho^-], \quad (98)$$

$$\sigma^+ = \frac{a_1(\rho^+)^2}{1 + a_1\rho^+}, \quad \sigma^- = \frac{(a_1 + a_2\rho^-)\rho_2^+}{1 + a_2\rho_2^+}. \quad (99)$$

These equations make it possible to compute recursively the series ρ^+ , ρ^- , η^+ , η^- , σ^+ and σ^- , knowing $W_{\mathcal{F}_{0,1}}^+$ and $W_{\mathcal{F}_{0,1}}^-$. The dissymmetry formula (25) will then yield $W_{\mathcal{B}}$, knowing $W_{\mathcal{F}}$:

$$\begin{aligned} W_{\mathcal{B}}(\mathbf{a}; \mathbf{b}; \mathbf{c}) &= W_{\mathcal{F}}[a_1; \rho^+; \rho^-] + W_{\mathcal{C}}[a_1; \rho^+ - \sigma^+; \rho^- - \sigma^-] \\ &\quad + \frac{a_1^2}{2} \left(\rho^+ - (\eta^+ + \sigma^+)(\rho^+ + 1) - \frac{1}{2}(\sigma_2^+ - (\sigma^+)^2) \right) \\ &\quad + \frac{a_2}{2} \left(\rho^- - (\eta^- + \sigma^-)(\rho^- + 1) - \frac{1}{2}(\sigma_2^- - (\sigma^-)^2) \right), \end{aligned} \quad (100)$$

where $W_{\mathcal{C}}$ is given by (31). If only the tilde generating functions are desired, for the unlabelled enumeration, equations (96 – 99) yield the following:

$$\tilde{\mathcal{R}}(x, y) = (1 + y) \exp\left(\sum_{i=1}^{\infty} \frac{(\mathcal{H} + \mathcal{S})^{\sim}(x^i, y^i)}{i}\right) - 1, \quad (101)$$

$$\tilde{\mathcal{R}}_{\tau}(x, y) = (1 + y) \exp\left(\sum_{i \text{ even}} \frac{(\mathcal{H} + \mathcal{S})^{\sim}(x^i, y^i)}{i} + \sum_{i \text{ odd}} \frac{(\mathcal{H} + \mathcal{S})_{\tau}^{\sim}(x^i, y^i)}{i}\right) - 1, \quad (102)$$

$$\tilde{\mathcal{H}}(x, y) = W_{\mathcal{F}_{0,1}}^+[x; \tilde{\mathcal{R}}(x, y); \tilde{\mathcal{R}}_{\tau}(x, y)], \quad (103)$$

$$\tilde{\mathcal{H}}_{\tau}(x, y) = W_{\mathcal{F}_{0,1}}^-[x; \tilde{\mathcal{R}}(x, y); \tilde{\mathcal{R}}_{\tau}(x, y)], \quad (104)$$

where the notation of (49) is used, and

$$\tilde{\mathcal{S}}(x, y) = \frac{x\tilde{\mathcal{R}}^2(x, y)}{1 + x\tilde{\mathcal{R}}(x, y)}, \quad (105)$$

$$\tilde{\mathcal{S}}_\tau(x, y) = \frac{(x + x^2\tilde{\mathcal{R}}_\tau(x, y))\tilde{\mathcal{R}}(x^2, y^2)}{1 + x^2\tilde{\mathcal{R}}(x^2, y^2)}. \quad (106)$$

Finally, equation (100) gives the following dissymmetry formula:

$$\begin{aligned} \tilde{\mathcal{B}}(x, y) &= W_{\mathcal{F}}[x; \tilde{\mathcal{R}}; \tilde{\mathcal{R}}_\tau] + W_{\mathcal{C}}[x; \tilde{\mathcal{R}} - \tilde{\mathcal{S}}; \tilde{\mathcal{R}}_\tau - \tilde{\mathcal{S}}_\tau] \\ &\quad + \frac{x^2}{2} \left(\tilde{\mathcal{R}} - (\tilde{\mathcal{H}} + \tilde{\mathcal{S}})(\tilde{\mathcal{R}} + 1) - \tilde{\mathcal{S}}_2 + \frac{1}{2}\tilde{\mathcal{S}}^2 \right) \\ &\quad + \frac{x^2}{2} \left(\tilde{\mathcal{R}}_\tau - (\tilde{\mathcal{H}}_\tau + \tilde{\mathcal{S}}_\tau)(\tilde{\mathcal{R}}_\tau + 1) + \frac{1}{2}\tilde{\mathcal{S}}_\tau^2 \right), \end{aligned} \quad (107)$$

where $\tilde{\mathcal{S}}_2(x, y) = \tilde{\mathcal{S}}(x^2, y^2)$.

5.1 3-connected graphs

The first application of the above formulas is the enumeration of unlabelled 3-connected graphs in 1982 (see [25], [18]). In this case, where $\mathcal{F} = \mathcal{F}_a$ and $\mathcal{B} = \mathcal{B}_a$, the species of all 3-connected and all 2-connected (simple) graphs, respectively, it is possible to compute the edge index series $W_{\mathcal{B}}$ directly, going from all graphs to connected graphs and then to 2-connected graphs. Since $\mathcal{R} = (1 + y)\mathcal{B}_{0,1} - 1$, the edge index series $\rho^+ = W_{\mathcal{R}}^+$ and $\rho^- = W_{\mathcal{R}}^-$ can be readily computed, as well as $\sigma^+ = W_{\mathcal{S}}^+$ and $\sigma^- = W_{\mathcal{S}}^-$, using (99), and then $\eta^+ = W_{\mathcal{H}}^+$ and $\eta^- = W_{\mathcal{H}}^-$, recursively, using equations (96) and (97). The edge index series $W_{\mathcal{F}}$ is then extracted recursively, using the dissymmetry formula (100), and the generating function $\tilde{\mathcal{F}}(x, y)$ is then immediately deduced. In [18], the computations are greatly simplified by introducing two auxiliary series $\beta(x, y)$ and $\gamma(x, y)$ satisfying

$$\rho^+[x, \beta(x, y), \gamma(x, y)] = y \quad \rho^-[x, \beta(x, y), \gamma(x, y)] = y. \quad (108)$$

The readers are referred to [18] for more details. See [25] and [18] for tables.

5.2 Series-parallel graphs and networks

At the other extreme lies the case with $\mathcal{F} = 0$, where the corresponding species of 2-connected graphs is the class $\mathcal{B} = \mathcal{G}_{\text{sp}}$ of *series-parallel graphs*. Thus, a series-parallel graph G is a 2-connected graph all of whose 3-components are polygons. G can also be characterized by the fact it contains no subdivision of K_4 .

The corresponding species of networks is the class $\mathcal{R} = \mathcal{R}_{\text{sp}}$ of *series-parallel networks*. An example of a series-parallel network is given in Figure 1. This class \mathcal{R} is defined recursively by the functional equation

$$\mathcal{R} = (1 + y)E\left(\frac{X\mathcal{R}^2}{1 + X\mathcal{R}}\right) - \mathbb{1}, \quad (109)$$

which is a specialization of equation (19) with $\mathcal{F} = 0$. We set $\rho^+ = W_{\mathcal{R}}^+(\mathbf{a}; \mathbf{b}; \mathbf{c})$ and $\rho^- = W_{\mathcal{R}}^-(\mathbf{a}; \mathbf{b}; \mathbf{c})$. Equations (96) and (97) then imply the following:

Corollary 20 *For the edge index series ρ^+ and ρ^- of the species $\mathcal{R} = \mathcal{R}_{\text{sp}}$ of series-parallel networks, we have the system of equations*

$$\rho^+ = (1 + b_1) \exp\left(\sum_{i=1}^{\infty} \frac{1}{i} \frac{a_i(\rho_i^+)^2}{1 + a_i\rho_i^+}\right) - 1, \quad (110)$$

$$\rho^- = (1 + c_1) \exp\left(\sum_{i \text{ even}} \frac{1}{i} \frac{a_i(\rho_i^+)^2}{1 + a_i\rho_i^+} + \sum_{i \text{ odd}} \frac{1}{i} \frac{(a_i + a_{2i}\rho_i^-)\rho_{2i}^+}{1 + a_{2i}\rho_{2i}^+}\right) - 1. \quad (111)$$

These edge index series can be computed recursively and the edge index series $W_{\mathcal{G}_{\text{sp}}}$ of series-parallel graphs can be deduced by specializing equation (100):

$$\begin{aligned} W_{\mathcal{G}_{\text{sp}}}(\mathbf{a}; \mathbf{b}; \mathbf{c}) &= W_{\mathcal{C}}[a_1; \rho^+ - \sigma^+; \rho^- - \sigma^-] + \frac{a_1^2}{2}(\rho^+ - \sigma^+ - \sigma^+\rho^+ - \frac{1}{2}(\sigma_2^+ - (\sigma^+)^2)) \\ &\quad + \frac{a_2}{2}(\rho^- - \sigma^- - \sigma^-\rho^- - \frac{1}{2}(\sigma_2^+ - (\sigma^-)^2)), \end{aligned} \quad (112)$$

where σ^+ and σ^- are defined by (99). The exponential and tilde generating functions are then immediately obtained. For example, for unlabelled series-parallel graphs counted according to the number of vertices (where we set $y = 1$), we find

$$\begin{aligned} \widetilde{\mathcal{G}}_{\text{sp}}(x) &= x^2 + x^3 + 2x^4 + 5x^5 + 15x^6 + 51x^7 + 230x^8 + 1142x^9 \\ &\quad + 6369x^{10} + 37601x^{11} + 232259x^{12} + 1476120x^{13} + 9599522x^{14} + \dots \end{aligned} \quad (113)$$

Similarly, for unlabelled series-parallel networks counted according to the number of internal vertices, we find that

$$\begin{aligned} \widetilde{\mathcal{R}}_{\text{sp}}(x) &= 1 + 2x + 8x^2 + 38x^3 + 208x^4 + 1220x^5 + 7592x^6 + 49006x^7 \\ &\quad + 325686x^8 + 2212112x^9 + 15290182x^{10} + 107191458x^{11} + 760349722x^{12} \\ &\quad + 5447100396x^{13} + 39354320204x^{14} + \dots, \end{aligned} \quad (114)$$

and for those that are τ -symmetric, we have

$$\begin{aligned} \widetilde{\mathcal{R}}_{\text{sp}\tau}(x) &= 1 + 2x + 4x^2 + 10x^3 + 24x^4 + 64x^5 + 168x^6 + 458x^7 + 1250x^8 + 3492x^9 \\ &\quad + 9734x^{10} + 27582x^{11} + 78078x^{12} + 223644x^{13} + 639948x^{14} + \dots \end{aligned} \quad (115)$$

Comparing with the existing literature on series-parallel networks, recall that here we are considering 01-networks with no parallel edges.

5.3 2-connected planar graphs and strongly planar networks.

Denote by \mathcal{F}_P the species of 3-connected planar graphs. Then, the corresponding species $\mathcal{B}(\mathcal{F}_P) = \mathcal{B}_P$ is the class of 2-connected planar graphs, and $\mathcal{R}(\mathcal{F}_P) = \mathcal{N}_P$ is the class of

strongly planar networks, that is non-trivial networks N such that $N \cup 01$ is planar. As before, we have

$$\mathcal{N}_P = (1 + y)(\mathcal{B}_P)_{01} - \mathbb{1}.$$

We compute all the edge index series and generating functions of these species up to 14 vertices. Previous results for unlabelled 2-connected planar graphs, up to 11 vertices by the number of vertices and up to 9 vertices and 12 edges by the number of vertices and edges, can be found in [15].

The enumeration of unlabelled planar graphs is one of the classical fundamental open problems in graph theory and combinatorics. A bridge between planar graph enumeration and planar map enumeration is provided by the fact that a 3-connected planar graph admits a unique embedding on the sphere up to homeomorphisms that either preserve or reverse the orientation of the sphere, which implies that any graph automorphism of a 3-connected planar graph is a map automorphism of the corresponding unsensed map, and conversely (see [26]).

In order to obtain the desired series, we used the program *Plantri* [3] to generate all the 3-connected planar maps (alias plane graphs) up to 14 vertices. In fact, version 4.2 of Plantri contains an option which yields the graphs together with all their automorphisms, and we computed their edge cycle indices, both cylindrical and Möbius. This way, the edge index series $W_{\mathcal{F}_P}$ of 3-connected planar graphs was secured up to 14 vertices. This yields successively the edge index series W^+ and W^- for the species $(\mathcal{F}_P)_{01}$, $\mathcal{R}(\mathcal{F}_P) = \mathcal{N}_P$, \mathcal{S} and \mathcal{H} (recursively), and, eventually, the edge index series $W_{\mathcal{B}_P}$ of 2-connected planar graphs, using the formulas of Section 5.0.2. The corresponding generating functions are immediately deduced. Thus, we extended up to 14 vertices the generating function $\widetilde{\mathcal{B}}_P(x, y)$ of unlabelled 2-connected planar graphs and up to 12 internal vertices the generating functions $\widetilde{\mathcal{N}}_P(x, y)$ and $\widetilde{\mathcal{N}}_{P\tau}(x, y)$ of unlabelled strongly planar networks. The coefficients of $\widetilde{\mathcal{B}}_P(x, y)$ are given in Table 1. Setting $y = 1$, we have

$$\begin{aligned} \widetilde{\mathcal{B}}_P(x) &= x^2 + x^3 + 3x^4 + 9x^5 + 44x^6 + 294x^7 + 2893x^8 + 36496x^9 + 545808x^{10} \\ &\quad + 9029737x^{11} + 159563559x^{12} + 2952794985x^{13} + 56589742050x^{14} + \dots \end{aligned} \quad (116)$$

$$\begin{aligned} \widetilde{\mathcal{N}}_P(x) &= 1 + 2x + 10x^2 + 72x^3 + 696x^4 + 8530x^5 + 124926x^6 \\ &\quad + 2068888x^7 + 37204942x^8 + 708076350x^9 + 14038364914x^{10} \\ &\quad + 287091103062x^{11} + 6016760068874x^{12} + \dots \end{aligned} \quad (117)$$

and

$$\begin{aligned} \widetilde{\mathcal{N}}_{P\tau}(x) &= 1 + 2x + 6x^2 + 20x^3 + 96x^4 + 470x^5 + 3074x^6 + 23408x^7 + 243482x^8 \\ &\quad + 3221018x^9 + 51729286x^{10} + 929983374x^{11} + 17911049418x^{12} + \dots \end{aligned} \quad (118)$$

Remark. Finding the edge index series for 3-connected planar maps without generating them is still an open problem. Wormald enumerated planar maps up to homeomorphism (see [27, 28] for 1-connected maps) without using cycle indices.

5.4 $K_{3,3}$ -free 2-connected graphs

A graph is called $K_{3,3}$ -free if it contains no subdivision of $K_{3,3}$ or, equivalently, if it has no minor isomorphic to $K_{3,3}$. As mentioned in the introduction, a theorem of Wagner [23] and Kelmans [12] implies that, if we take $\mathcal{F} = \mathcal{F}_P + K_5$, then the corresponding species $\mathcal{B} = \mathcal{B}(\mathcal{F})$ of 2-connected graphs all of whose 3-connected components are planar

n	m	$g_{n,m}$	n	m	$g_{n,m}$	n	m	$g_{n,m}$
2	1	1	10	10	1	13	13	1
3	3	1	10	11	9	13	14	15
4	4	1	10	12	121	13	15	428
4	5	1	10	13	1018	13	16	8492
4	6	1	10	14	5617	13	17	107771
5	5	1	10	15	20515	13	18	903443
5	6	2	10	16	52068	13	19	5287675
5	7	3	10	17	94166	13	20	22514501
5	8	2	10	18	123357	13	21	71869047
5	9	1	10	19	116879	13	22	175632924
6	6	1	10	20	79593	13	23	333410770
6	7	3	10	21	37859	13	24	496146048
6	8	9	10	22	12066	13	25	581318637
6	9	13	10	23	2306	13	26	536073583
6	10	11	10	24	233	13	27	386948719
6	11	5	11	11	1	13	28	216020293
6	12	2	11	12	11	13	29	91369743
7	7	1	11	13	189	13	30	28288016
7	8	4	11	14	2210	13	31	6047730
7	9	20	11	15	16650	13	32	797583
7	10	49	11	16	83105	13	33	49566
7	11	77	11	17	289532	14	14	1
7	12	75	11	18	727243	14	15	18
7	13	47	11	19	1347335	14	16	616
7	14	16	11	20	1861658	14	17	15350
7	15	5	11	21	1926664	14	18	243897
8	8	1	11	22	1485235	14	19	2550530
8	9	6	11	23	841152	14	20	18598574
8	10	40	11	24	339390	14	21	98777626
8	11	158	11	25	92751	14	22	394640925
8	12	406	11	26	15362	14	23	1214212848
8	13	662	11	27	1249	14	24	2926745166
8	14	737	12	12	1	14	25	5594151239
8	15	538	12	13	13	14	26	8546948259
8	16	259	12	14	292	14	27	10481908901
8	17	72	12	15	4476	14	28	10324262525
8	18	14	12	16	44297	14	29	8139338353
9	9	1	12	17	290680	14	30	5095275794
9	10	7	12	18	1333029	14	31	2497740781
9	11	70	12	19	4434175	14	32	937658515
9	12	426	12	20	10992850	14	33	260094850
9	13	1645	12	21	20663187	14	34	50215417
9	14	4176	12	22	29764598	14	35	6022143
9	15	7307	12	23	32990517	14	36	339722
9	16	8871	12	24	28087447			
9	17	7541	12	25	18199252			
9	18	4353	12	26	8814281			
9	19	1671	12	27	3088000			
9	20	378	12	28	740272			
9	21	50	12	29	108597			
			12	30	7595			

Table 1: The number $g_{n,m}$ of unlabelled 2-connected planar graphs having n vertices and m edges.

or isomorphic to K_5 is the class of $K_{3,3}$ -free 2-connected graphs. Computations are similar to those in the preceding Section 5.3. For example, we find that

$$\begin{aligned} \tilde{\mathcal{B}}(x) = & x^2 + x^3 + 3x^4 + 10x^5 + 46x^6 + 308x^7 + 2997x^8 + 37471x^9 + 556637x^{10} \\ & + 9171526x^{11} + 161679203x^{12} + 2987857791x^{13} + 57218439783x^{14} + \dots \end{aligned} \quad (119)$$

$$\begin{aligned} \tilde{\mathcal{R}}(x) = & 1 + 2x + 10x^2 + 74x^3 + 718x^4 + 8786x^5 + 128006x^6 \\ & + 2108610x^7 + 37767136x^8 + 716900760x^9 + 14191084858x^{10} \\ & + 289958295858x^{11} + 6074048514588x^{12} + \dots \end{aligned} \quad (120)$$

and

$$\begin{aligned} \tilde{\mathcal{R}}_\tau(x) = & 1 + 2x + 6x^2 + 22x^3 + 102x^4 + 518x^5 + 3362x^6 + 25890x^7 + 267988x^8 \\ & + 3524132x^9 + 56099830x^{10} + 1001483346x^{11} + 19189524860x^{12} + \dots \end{aligned} \quad (121)$$

5.5 $K_{3,3}$ -free projective-planar and toroidal graphs.

Denote by PP the species of projective-planar graphs which are $K_{3,3}$ -free, 2-connected and non-planar. In [6] we proved the following structural characterization of PP:

Theorem 21 ([6]) *The species PP of $K_{3,3}$ -free 2-connected non-planar projective-planar graphs can be expressed as a canonical composition*

$$\text{PP} = K_5 \uparrow \mathcal{N}_P, \quad (122)$$

where \mathcal{N}_P denotes the species of strongly planar networks. ■

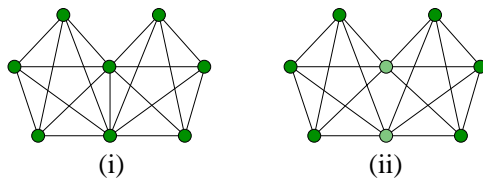


Figure 10: (i) The graph M , (ii) The graph M^* .

A similar characterization is provided in [7] for the species \mathcal{T} of $K_{3,3}$ -free 2-connected toroidal (non-planar) graphs. Given two disjoint K_5 -graphs, the graph obtained by identifying an edge of one of the K_5 's with an edge of the other is called an M -graph (see Figure 10 (i)), and, when the edge of identification is deleted, an M^* -graph (see Figure 10 (ii)).

A *toroidal crown* is a graph H obtained from an unoriented cycle C_i , $i \geq 3$, by substituting $(K_5)_{01}$ -networks for some edges of C_i in such a way that no two unsubstituted edges of C_i are adjacent in H (see Figure 11 (ii) for an example). Denote by \mathcal{H} the class of toroidal crowns. A *toroidal core* is defined as either K_5 , an M -graph, an M^* -graph, or a toroidal crown.

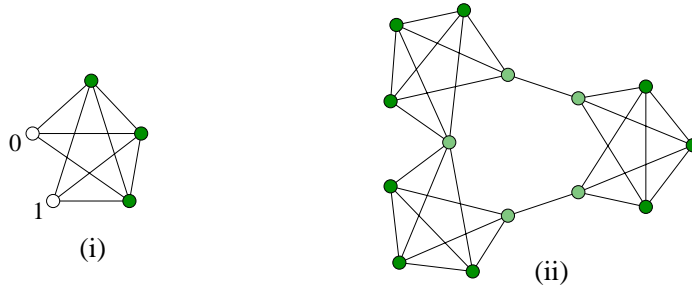


Figure 11: (i) A $(K_5)_{01}$ -network, (ii) A toroidal crown.

Theorem 22 ([7]) *The species \mathcal{T} of $K_{3,3}$ -free 2-connected non-planar toroidal graphs can be expressed as a canonical composition*

$$\mathcal{T} = \mathcal{T}_C \uparrow \mathcal{N}_P, \quad (123)$$

where \mathcal{T}_C denotes the class of toroidal cores, i.e. $\mathcal{T}_C = K_5 + M + M^* + \mathcal{H}$. ■

In [8] we give explicit formulas of the edge index series for the graphs K_5 , M , and M^* and for the species \mathcal{H} of toroidal crowns. In order to enumerate unlabelled graphs in PP or \mathcal{T} according to the number of vertices and edges, the generating functions $\widetilde{\mathcal{N}}_P(x, y)$ and $\widetilde{\mathcal{N}}_{P\tau}(x, y)$ are also required, because we have

$$\widetilde{\text{PP}}(x, y) = W_{K_5}[x; \widetilde{\mathcal{N}}_P(x, y); \widetilde{\mathcal{N}}_{P\tau}(x, y)], \quad (124)$$

$$\widetilde{\mathcal{T}}(x, y) = W_{\mathcal{T}_C}[x; \widetilde{\mathcal{N}}_P(x, y); \widetilde{\mathcal{N}}_{P\tau}(x, y)]. \quad (125)$$

Using the results of Section 5.3, we have extended previous tables up to 17 vertices for $\widetilde{\text{PP}}(x, y)$ and up to 20 vertices for $(\widetilde{\mathcal{T}} - \widetilde{\text{PP}})(x, y)$. Setting $y = 1$, we obtain

$$\begin{aligned} \widetilde{\text{PP}}(x) &= x^5 + 2x^6 + 14x^7 + 102x^8 + 962x^9 + 10662x^{10} \\ &\quad + 139764x^{11} + 2088482x^{12} + 34680722x^{13} + 622943224x^{14} \\ &\quad + 11854223815x^{15} + 235386309134x^{16} + 4826871283270x^{17} + \dots \end{aligned} \quad (126)$$

$$\begin{aligned} (\widetilde{\mathcal{T}} - \widetilde{\text{PP}})(x) &= 2x^8 + 11x^9 + 127x^{10} + 1388x^{11} + 16905x^{12} \\ &\quad + 214191x^{13} + 2890154x^{14} + 41748279x^{15} + 650024679x^{16} \\ &\quad + 10888386896x^{17} + 194674234840x^{18} + 3674322404851x^{19} \\ &\quad + 72412623360105x^{20} + \dots \end{aligned} \quad (127)$$

5.6 Homeomorphically irreducible graphs.

A graph is called *homeomorphically irreducible* if it contains no vertices of degree 2. In order to enumerate these graphs, we can apply the method of Walsh and Robinson ([18, 25]) as follows. Any 2-connected graph G is either a series-parallel graph or contains a unique 2-connected homeomorphically irreducible core $C(G)$, which is different from K_2 , and unique components $\{N_e\}_{e \in E(C(G))}$, which are series-parallel networks, whose composition

gives G . Let \mathcal{B} be a species of 2-connected graphs. Denote by $I_{\mathcal{B}}$ the class of graphs which are homeomorphically irreducible cores of graphs in \mathcal{B} . Also, set $\mathcal{B}_{\text{sp}} = \mathcal{B} \cap \mathcal{G}_{\text{sp}}$ which is the class of series-parallel graphs in \mathcal{B} , and denote by \mathcal{R}_{sp} the species of series-parallel networks. Then we have the following proposition.

Proposition 23 ([6, 24]) *Let \mathcal{B} be a species of 2-connected graphs such that*

1. $I_{\mathcal{B}}$ is contained in \mathcal{B} ,
2. \mathcal{B} is closed under edge substitution by series-parallel networks.

Then we have

$$\mathcal{B} = \mathcal{B}_{\text{sp}} + I_{\mathcal{B}} \uparrow \mathcal{R}_{\text{sp}}, \quad (128)$$

the composition $I_{\mathcal{B}} \uparrow \mathcal{R}_{\text{sp}}$ being canonical. ■

Proposition 24 ([18]) *There exist unique series $\beta(x, y)$ and $\gamma(x, y)$ satisfying*

$$\rho^+[x, \beta(x, y), \gamma(x, y)] = y, \quad \rho^-[x, \beta(x, y), \gamma(x, y)] = y, \quad (129)$$

where $\rho^+ = W_{\mathcal{R}_{\text{sp}}}^+$ and $\rho^- = W_{\mathcal{R}_{\text{sp}}}^-$. Moreover, these series are given explicitly by

$$\beta(x, y) = -1 + (1 + y) \prod_{j \geq 1} (1 - x^{2j-1} y^{2j}) (1 - x^{2j} y^{2j+1})^{-1}, \quad (130)$$

$$\gamma(x, y) = -1 + (1 + y) \prod_{j \geq 1} \frac{(1 - x^{4j-3} y^{4j-2})(1 + x^{4j-1} y^{4j})}{(1 + x^{4j-2} y^{4j-1})(1 - x^{4j} y^{4j+1})}. \quad (131)$$

Proof. We first note that

$$\begin{aligned} \rho^+ &= b_1 + a_1 b_1^2 + a_1 b_1^3 + \cdots, \\ \rho^- &= c_1 + a_1 b_2 + a_1 b_2 c_1 \cdots, \end{aligned}$$

where the remaining terms are of higher order in the vertex-cycle variables so that equations (129) determine recursively unique series $\beta(x, y)$ and $\gamma(x, y)$. In fact, from (110) and (111), we see that $\beta(x, y)$ and $\gamma(x, y)$ must satisfy

$$1 + y = (1 + \beta(x, y)) \exp \left(\sum_{i \geq 1} \frac{1}{i} \frac{x^i y^{2i}}{1 + x^i y^i} \right), \quad (132)$$

$$1 + y = (1 + \gamma(x, y)) \exp \left(\sum_{i \text{ even}} \frac{1}{i} \frac{x^i y^{2i}}{1 + x^i y^i} + \sum_{i \text{ odd}} \frac{1}{i} \frac{(x^i + x^{2i} y^i) y^{2i}}{1 + x^{2i} y^{2i}} \right), \quad (133)$$

from which (130) and (131) are readily deduced. ■

Proposition 25 ([18]) *For the species \mathcal{G}_{sp} of series parallel graphs, we have*

$$W_{\mathcal{G}_{\text{sp}}}[x; \beta(x, y); \gamma(x, y)] = -x^2 y^2 + xy(x + xy(1 - x))(1 - x^4 y^4)^{-1}. \quad (134)$$

Proof. Notice that for the edge index series $\sigma^+ = W_S^+$ and $\sigma^- = W_S^-$ of series-parallel s -networks, we have, using (99),

$$\sigma^+[x; \beta(x, y); \gamma(x, y)] = \frac{xy^2}{1 + xy}, \quad \sigma^-[x; \beta(x, y); \gamma(x, y)] = \frac{(x + x^2y)y^2}{1 + x^2y^2}. \quad (135)$$

It is then possible to use the dissymmetry formula (112), and the result follows after some simplifications. \blacksquare

Corollary 26 *Let \mathcal{B} be a species of 2-connected graphs such that the hypothesis of Proposition 23 are satisfied. Then we have*

$$\widetilde{I}_{\mathcal{B}}(x, y) = (W_{\mathcal{B}} - W_{\mathcal{B}_{\text{sp}}})[x; \beta(x, y); \gamma(x, y)], \quad (136)$$

where $\mathcal{B}_{\text{sp}} = \mathcal{B} \cap \mathcal{G}_{\text{sp}}$.

5.6.1 Example: Planar graphs

For the species $\mathcal{B} = \mathcal{B}_P$ of 2-connected planar graphs, we have $\mathcal{B} \cap \mathcal{G}_{\text{sp}} = \mathcal{G}_{\text{sp}}$. It follows from Proposition 25 and Corollary 26 that for the species I_P of 2-connected homeomorphically irreducible planar graphs, we have

$$\widetilde{I}_P(x, y) = W_{\mathcal{B}_P}[x; \beta(x, y); \gamma(x, y)] + x^2y^2 - xy(x + xy(1 - x))(1 - x^4y^4)^{-1}. \quad (137)$$

5.6.2 Example: $K_{3,3}$ -free 2-connected graphs

As seen in Section 5.4, the species $\mathcal{B} = \mathcal{B}_{\mathcal{F}}$ associated with the class $\mathcal{F} = \mathcal{F}_P + K_5$ consists of $K_{3,3}$ -free 2-connected graphs. Again, we have $\mathcal{G}_{\text{sp}} \subset \mathcal{B}$, and, for the species I of homeomorphically irreducible $K_{3,3}$ -free 2-connected graphs, we have

$$\widetilde{I}(x, y) = W_{\mathcal{B}}[x; \beta(x, y); \gamma(x, y)] + x^2y^2 - xy(x + xy(1 - x))(1 - x^4y^4)^{-1}. \quad (138)$$

5.6.3 Example: $K_{3,3}$ -free projective planar and toroidal graphs

For the species $\mathcal{B} = \text{PP}$ and $\mathcal{B} = \mathcal{T}$ of 2-connected $K_{3,3}$ -free (non-planar) projective planar and toroidal graphs, respectively, we have $\mathcal{B} \cap \mathcal{G}_{\text{sp}} = \emptyset$. It follows that for the corresponding species I_{PP} and $I_{\mathcal{T}}$ of homeomorphically irreducible graphs we have

$$\begin{aligned} \widetilde{I}_{\text{PP}}(x, y) &= W_{\text{PP}}[x; \beta(x, y); \gamma(x, y)] \\ &= W_{K_5 \uparrow \mathcal{N}_P}[x; \beta(x, y); \gamma(x, y)] \\ &= W_{K_5} [x; W_{\mathcal{N}_P}^+[x; \beta(x, y); \gamma(x, y)]; W_{\mathcal{N}_P}^- [x; \beta(x, y); \gamma(x, y)]] \end{aligned} \quad (139)$$

and

$$\begin{aligned} \widetilde{I}_{\mathcal{T}}(x, y) &= W_{\mathcal{T}}[x; \beta(x, y); \gamma(x, y)] \\ &= W_{\mathcal{T}_C} [x; W_{\mathcal{N}_P}^+[x; \beta(x, y); \gamma(x, y)]; W_{\mathcal{N}_P}^- [x; \beta(x, y); \gamma(x, y)]] . \end{aligned} \quad (140)$$

References

- [1] F. Bergeron, G. Labelle, P. Leroux, *Combinatorial Species and Tree-Like Structures*, Cambridge Univ. Press, Cambridge, 1998.
- [2] E.A. Bender, Zh. Gao, N.C. Wormald, The number of labeled 2-connected planar graphs, *Electron. J. Combin.* 9 (1) (2002), research paper 43, 13 pp. (electronic).
- [3] G. Brinkmann, B. McKay, The computer software “plantri and fullgen”, <http://cs.anu.edu.au/~bdm/plantri/>, 2001.
- [4] W.H. Cunningham, J. Edmonds, A combinatorial decomposition theory, *Canad. J. Math.* 32 (3) (1980) 734–765.
- [5] T. Fowler, I. Gessel, G. Labelle, P. Leroux, The specification of 2-trees, *Adv. in Appl. Math.* 28 (2) (2002) 145–168.
- [6] A. Gagarin, G. Labelle, P. Leroux, The structure and labelled enumeration of $K_{3,3}$ -subdivision-free projective-planar graphs, *Pure Math. Appl.* 16 (3) (2005) 267–286.
- [7] A. Gagarin, G. Labelle, P. Leroux, The structure of $K_{3,3}$ -subdivision-free toroidal graphs, *Discrete Math.* 307 (23) (2007) 2993–3005.
- [8] A. Gagarin, G. Labelle, P. Leroux, Counting unlabelled toroidal graphs with no $K_{3,3}$ -subdivisions, *Adv. in Appl. Math.* 39 (1) (2007) 51–75.
- [9] O. Giménez, M. Noy, J.J. Rué, Graph classes with given 3-connected components: asymptotic counting and critical phenomena, *Electron. Notes Discrete Math.* 29 (2007) 521–529.
- [10] J.E. Hopcroft, R.E. Tarjan, Dividing a graph into triconnected components, *SIAM J. Comput.* 2 (1973) 135–158.
- [11] A. Joyal, Une théorie combinatoire des séries formelles, *Adv. in Math.* 42 (1) (1981) 1–82.
- [12] A.K. Kelmans, Graph expansion and reduction, in: *Algebraic methods in graph theory*, Vol. I (Szeged, 1978), *Colloq. Math. Soc. János Bolyai*, 25, North-Holland, Amsterdam-New York, 1981, 317–343.
- [13] G. Labelle, C. Lamathe, P. Leroux, Labelled and unlabelled enumeration of k -gonal 2-trees, *J. Combin. Theory Ser. A* 106 (2) (2004) 193–219.
- [14] S. MacLane, A structural characterization of planar combinatorial graphs, *Duke Math. J.* 3 (3) (1937) 460–472.
- [15] B. McKay, Sequences A021103 and A049336, in: N.J.A. Sloane (Ed.), *The On-Line Encyclopedia of Integer Sequences*, <http://www.research.att.com/~njas/sequences/>, 2009.

- [16] R. Otter, The number of trees, *Ann. of Math.* 49 (1948) 583–599.
- [17] R.W. Robinson, Enumeration of non-separable graphs, *J. Combin. Theory* 9 (1970) 327–356.
- [18] R.W. Robinson, T.R. Walsh, Inversion of cycle index sum relations for 2- and 3-connected graphs, *J. Combin. Theory Ser. B* 57 (2) (1993) 289–308.
- [19] R. Thomas, Recent excluded minor theorems for graphs, in: *Surveys in Combinatorics*, London Math. Soc. Lecture Note Ser., 267, Cambridge Univ. Press, Cambridge, 1999, 201–222.
- [20] B.A. Trakhtenbrot, The theory of non-repeating contact schemes, *Trudy Mat. Inst. Steklov.* 51 (1958) 226–269 (in Russian).
- [21] W.T. Tutte, A theory of 3-connected graphs, *Nederl. Akad. Wetensch. Proc. Ser. A* 64 (1961) 441–455.
- [22] W.T. Tutte, Graph Theory, *Encyclopedia of Mathematics and its Applications*, vol. 21, Addison-Wesley Publishing Company, Reading, MA, 1984.
- [23] K. Wagner, Über eine Erweiterung eines Satzes von Kuratowski, *Deutsche Math.* 2 (1937) 280–285 (in German).
- [24] T.R.S. Walsh, Counting labelled three-connected and homeomorphically irreducible two-connected graphs, *J. Combin. Theory Ser. B* 32 (1) (1982) 1–11.
- [25] T.R.S. Walsh, Counting unlabelled three-connected and homeomorphically irreducible two-connected graphs, *J. Combin. Theory Ser. B* 32 (1) (1982) 12–32.
- [26] H. Whitney, 2-Isomorphic graphs, *Amer. J. Math.* 55 (1–4) (1933) 245–254.
- [27] N.C. Wormald, On the number of planar maps, *Canad. J. Math.* 33 (1) (1981) 1–11.
- [28] N.C. Wormald, Counting unrooted planar maps, *Discrete Math.* 36 (2) (1981) 205–225.