

# Weak Near-Unanimity Functions and NP-Completeness Proofs

Cobus Swarts  
joint work with Gary MacGillivray



University of Victoria

The 2009 Canadian Discrete and  
Algorithmic Mathematics Conference  
May 28, 2009

- The Homomorphism Problem.
  - Complexity in the undirected case: Hell and Nešetřil (1990).
  - Complexity in the directed case: **Unknown**, only special cases e.g. semi-complete digraphs (Bang-Jensen, Hell and MacGillivray, 1988), smooth digraphs (Barto, Kozik and Niven 2008).
- Weak Near-Unanimity Functions and NP-completeness.
  - Conjecture of Bulatov, Jeavons and Krokhin.
  - Gather evidence for this conjecture:
    - Some polynomial cases:  $\underline{X}$ ,  $C_k$ -extended  $\underline{X}$ , graft extension.
    - NP-completeness: Indicator construction, vertex and arc sub-indicators.
- WNUFs/no WNUFs for undirected graphs, semi-complete digraphs, and vertex transitive digraphs.

# The Homomorphism Problem

- Let  $G$  and  $H$  be digraphs. A **homomorphism** from  $G$  to  $H$  is a mapping  $f : V(G) \rightarrow V(H)$  such that  $xy \in A(G)$  implies that  $f(x)f(y) \in A(H)$ . The existence of such a homomorphism is denoted by  $G \rightarrow H$ .
- The homomorphism problem for the digraph  $H$  is the problem of deciding for a given input  $G$  whether  $G \rightarrow H$ . This is also known as the  **$H$ -colouring problem**, and is denoted by  **$\text{HOM}_H$** .

# Complexity in the Undirected Case

In the undirected case there is a **dichotomy**:

**Theorem (Hell and Nešetřil 1990)**

*Let  $H$  be a graph with loops allowed.*

- *If  $H$  is bipartite or contains a loop, then the  $H$ -colouring problem has a polynomial time algorithm.*
- *Otherwise the  $H$ -colouring problem is NP-complete.*

# Complexity in the Directed Case

Only known in special cases:

## Theorem (Bang-Jensen, Hell and MacGillivray 1988)

*Let  $H$  be a semi-complete digraph.*

- *If  $H$  contains at most one directed cycle, then  $H$ -colouring is polynomial time solvable.*
- *Otherwise  $H$ -colouring is NP-complete.*

## Theorem (Barto, Kozik and Niven 2008)

*Let  $H$  be a digraph with no sources and no sinks.*

- *If the core of  $H$  is a directed cycle, then  $\text{HOM}_H$  is polynomial.*
- *If the core of  $H$  is not a directed cycle, then  $\text{HOM}_H$  is NP-complete.*

# Weak Near-Unanimity Functions

The result by Barto, Kozik and Niven uses tools from universal algebra:

**Theorem (Bulatov, Jeavons and Krokhin; Larose and Zádori; Maróti and McKenzie)**

*If the digraph  $H$  does not admit a weak near unanimity function, then  $\text{HOM}_H$  is NP-complete.*

The existence of such a function is conjectured by Bulatov, Jeavons and Krokhin to determine the complexity of  $\text{HOM}_H$  exactly:

**Conjecture**

*If the digraph  $H$  admits a WNUF, then  $\text{HOM}_H$  is polynomial, otherwise (if  $H$  does not admit a WNUF)  $\text{HOM}_H$  is NP-complete.*

# Weak Near-Unanimity Functions

A weak near-unanimity function  $f$  of arity  $k$  ( $\text{WNUF}_k$ ) on the digraph  $H$  is

- a **polymorphism**:  $f : H^k \rightarrow H$ ,

$$V(H^k) = \overbrace{V(H) \times V(H) \times \cdots \times V(H)}^k$$
$$(x_1, x_2, \dots, x_k) \rightarrow (y_1, y_2, \dots, y_k) \text{ iff } x_i \rightarrow y_i, 1 \leq i \leq k,$$

- **idempotent**:  $f(x, x, \dots, x) = x$ ,
- **weakly nearly-unanimous**:

$$f(y, x, x, \dots, x, x) = f(x, y, x, \dots, x, x) = f(x, x, y, \dots, x, x) = \cdots = f(x, x, x, \dots, x, y).$$

# Conjecture of Bulatov, Jeavons and Krokhin

## Conjecture (Bulatov, Jeavons and Krokhin)

*If  $H$  admits a WNUF, then  $\text{HOM}_H$  is polynomial time solvable.  
Otherwise  $\text{HOM}_H$  is NP-complete (this part is known).*

To gather evidence for this conjecture we could:

- Show that known algorithmic methods  $\Rightarrow$  WNUFs.
- Try to show that digraphs  $H$  for which  $\text{HOM}_H$  is NP-complete, have no WNUF. Try to prove “no-WNUF” theorems.
- Prove it for various graph families.
  - Good candidates are ones where a dichotomy is known.

## Theorem (Gutjahr, Woeginger and Welzl 1992)

*Let  $H$  be a digraph such that*

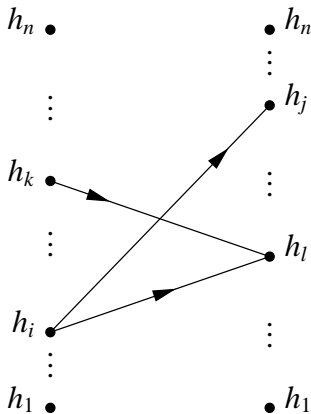
- *$H$  has an  $\underline{X}$ -enumeration, or*
- *$H$  has the  $C_k$ -extended  $\underline{X}$  property, or*
- *$H$  is an instance of the graft extension.*

*Then  $\text{HOM}_H$  is polynomial time solvable.*

The goal here is to show that if a digraph  $H$  has any of these properties, then it also has a WNUF.

# The $\underline{X}$ -enumeration

Let  $\{h_1, h_2, \dots, h_n\}$  be an enumeration of the vertices of a digraph  $H$ . This enumeration is said to satisfy the  $\underline{X}$  property if  $h_i h_j, h_k h_l \in A(H)$ , implies that  $\min(h_i, h_k) \min(h_j, h_l) \in A(H)$ , where the minimum is taken with respect to the enumeration of  $V(H)$ .



# The $\underline{X}$ -enumeration and WNUFs

## Theorem

If  $H$  has an  $\underline{X}$ -enumeration, then  $H$  has a  $\text{WNUF}_2$ .

**Proof:** Define  $f : H^2 \rightarrow H$  by  $f(x_1, x_2) = \min\{x_1, x_2\}$ . ■

A converse is also true.

## Theorem

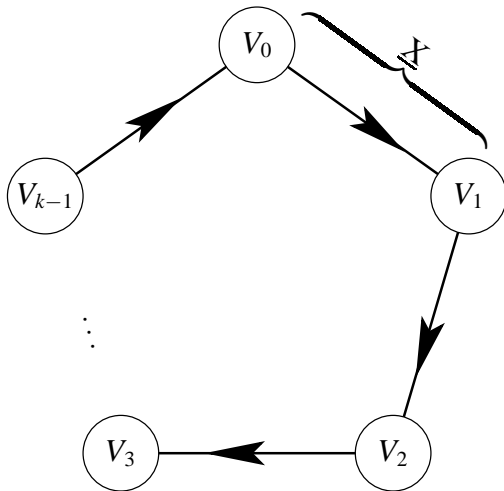
If there is an enumeration of  $V(H)$  and a  $\text{WNUF}_2$ ,  $f : H^2 \rightarrow H$ , such that for all  $x_1, x_2 \in V(H)$ ,  $f(x_1, x_2) = \min\{x_1, x_2\}$ , then this enumeration is an  $\underline{X}$ -enumeration.

**Proof:** Suppose  $u_1u_2$  and  $v_1v_2 \in E(H)$ . Then

$(u_1, v_1)(u_2, v_2) \in E(H \times H)$ , so  $f(u_1, v_1) = \min\{u_1, v_1\}$  is adjacent to  $f(u_2, v_2) = \min\{u_2, v_2\}$  in  $H$ . ■

Arity 2 can be replaced by arity  $k$ .

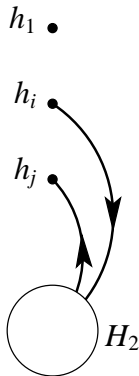
# The $C_k$ -extended $\underline{X}$ Property



# The $\underline{X}$ -graft Extension

- $h_1, h_2, \dots, h_n$  is an  $\underline{X}$ -enumeration  $H_1$ .
- $H_2$  is another digraph.

Form  $H$  by replacing  $h_n$  by  $H_2$  as in the wreath product. The digraph  $H$  is called  $\text{graft}(H_1, H_2)$ .



## Theorem

Let  $H$  be a digraph such that

- $H$  has an X-enumeration, or
- $H$  has the  $C_k$ -extended X property, or
- $H = \text{graft}(H_1, H_2)$ , where  $H_2$  has a  $\text{WNUF}_k$ .

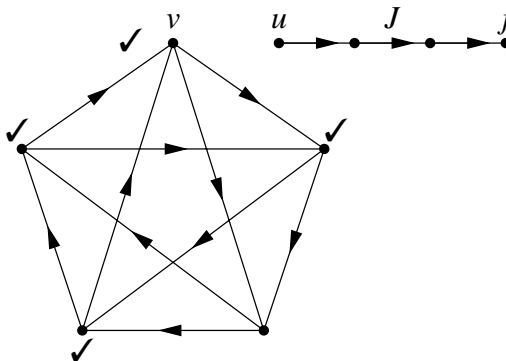
Then  $H$  has a  $\text{WNUF}_k$ .

Four main tools in proving NP-completeness of  $\text{HOM}_H$ :

- Direct reduction from SAT or similar.
- The “indicator” construction.
- The “(vertex) sub-indicator” construction.
- The “arc sub-indicator” construction.

For digraphs  $H$  in some family, the proofs normally go by induction on  $|V(H)|$  (non-existence of minimum counterexample). The direct reductions establish the base cases and the three constructions make the induction work.

# Vertex Sub-indicator Example



Identify  $u$  and  $v$ , retract  $J$  back to  $T_5$ , and keep track of the images of  $j$ .

# Vertex Sub-indicator

In general, some designated vertices of the sub-indicator  $J$  are identified with some designated vertices of  $H$ . Let  $H^+$  be the subgraph of  $H$  induced by the images of  $j$  under retraction back to  $H$ .

**Lemma (Hell and Nešetřil 1990)**

*Let  $H$  be a digraph that is a core. If the  $H^+$ -colouring problem is NP-complete, then the  $H$ -colouring problem is also NP-complete.*

**Lemma**

*Let  $H$  be a digraph. If  $H^+$  does not have a  $\text{WNUF}_k$  for  $k > 1$ , then  $H$  does not have a  $\text{WNUF}$  of arity greater than one.*

The proof shows the contrapositive.

There are similar results corresponding to the indicator construction and arc sub-indicator construction.

# NP-completeness Proofs and WNUFs

These results give a method for translating certain NP-completeness theorems into no WNUF theorems.

- “NP-complete”  $\rightsquigarrow$  “no WNUF”
- If there is a proof of NP-completeness that depends on base cases  $B_1, B_2, \dots, B_t$  and the application of vertex (arc) sub-indicators and indicators, then there is a proof of “no WNUF” provided one can show that  $B_1, B_2, \dots, B_t$  have no WNUF.

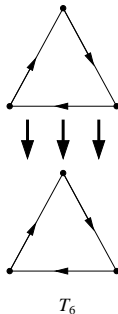
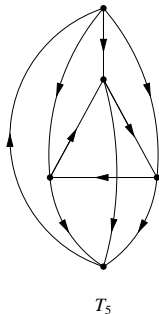
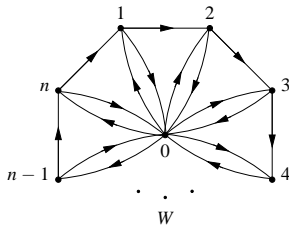
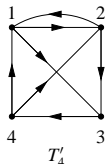
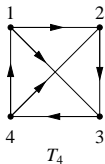
## Theorem (Bang-Jensen, Hell and MacGillivray 1988)

*Let  $H$  be a semi-complete digraph.*

- *If  $H$  contains at most one directed cycle, then  $H$ -colouring is polynomial time solvable.*
  - *Otherwise  $H$ -colouring is NP-complete.*
- 
- Acyclic tournaments have an  $\underline{X}$ -enumeration  $\Rightarrow$  have a WNUF.
  - Unicyclic tournaments are instances of the graft extension  $\Rightarrow$  have a WNUF.

# Semi-complete Digraphs With At Least Two Cycles

Base Cases:



## Theorem

*Let  $H$  be a semi-complete digraph.*

- *If  $H$  contains at most one directed cycle, then  $H$  has a WNUF.*
- *Otherwise  $H$  does not admit a WNUF.*

## Theorem (Hell and Nešetřil 1990)

*Let  $H$  be a core.*

- *If  $H$  is bipartite, then the  $H$ -colouring problem has a polynomial time algorithm.*
- *Otherwise the  $H$ -colouring problem is NP-complete.*

## Theorem

*Let  $H$  be a core.*

- *If  $H = K_2$ , then  $H$  admits a  $NUF_3$ .*
- *If  $H$  is non-bipartite, then  $H$  does not admit a WNUF.*

Base case is  $K_3$ .

# Vertex Transitive Digraphs and WNUFs

## Theorem (MacGillivray 1991)

*Let  $H$  be a vertex transitive digraph that is also a core.*

- *If  $H = C_k$ ,  $HOM_H$  is polynomial.*
- *Otherwise,  $HOM_H$  is NP-complete.*

## Theorem

*Let  $H$  be a vertex transitive digraph that is also a core.*

- *If  $H = C_k$ , then  $H$  admits a  $NUF_3$ .*
- *Otherwise,  $H$  does not admit a WNUF.*

Base Cases: Undirected non-bipartite graphs.

Thank you.  
Questions?