Packing via Covering and LP-Relative Approximation

Jochen Könemann, Ojas Parekh, *David Pritchard*

Department of Combinatorics & Optimization, U Waterloo, ON

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Linear Program-Based Combinatorial Optimization in a Nutshell

- LPs used for exact algorithms since mid-century (e.g., flow, matching, matroids)
  - Literature developed lots of *proof techniques & algorithmic techniques* (LP duality, total unimodularity, uncrossing, ellipsoid algorithm)
- Broad base of knowledge applied also to <u>approximation</u> algorithms ~30 years ago
  - Spurred more techniques, e.g. primal-dual schema, randomized rounding, scaling, grouping, iterated rounding/relaxation

Technique-Based Theory

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- LP-based combinatorial optimization thereby exhibits the best & worst of mathematical problem-solving:
  - Best: techniques are elementary when considered individually, and combining them gives some very great results
  - Worst: it can be very hard to figure out how to combine them!

Approximation algorithms

- For NP-hard optimization problems optimal value OPT cannot be found by a poly-time algorithm (unless P=NP)
- Thus in poly-time the best we can do is find an *approximately* optimal answer
- Our convention: An α-*approximation algorithm* for a "max" problem always returns feasible solution with value at least OPT/α. For "min" problem, always returns value at most OPT*α. ($\alpha \geq 1$; exact ≡ $\alpha = 1$)

Iterated LP Relaxation

- Background: Iterated relaxation (Lau, Naor, Salavatipour, Singh, STOC 2007) finds solutions with *additive violation of constraints* instead of multiplicative factor
  - Bounded degree spanning tree: can find a spanning tree with super-optimal weight but violating degree bounds by up to +1 (SL '07)
  - Integral multicommodity flow in a tree: can find a flow with super-optimal weight but violating edge capacities by up to +2 (KPP '08)

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- Can iterated rounding help when constraints are inflexible?
- Our contribution: Techniques that remove violation (at expense of value)
  - General approach that works in most iterated rounding situations (where a “counting lemma” exists)
- Gives new algorithmic results
  - Ultimate goal: understanding best possible approximation ratios for problems

Rest of Talk

- Def^n: Multicommodity Flow in a Tree (MFT)
- Def^n: LP-Relative Approximation Ratio
- Counting lemma for MFT and consequences
- Results in case of MFT:
  - 1+1/O(*minimum capacity*) approx algorithms
- Mini-Technique: LP-Relative Covers Help Cover
- Technique: LP-Relative Covers Help Pack
- General Form of Results, Open Problems

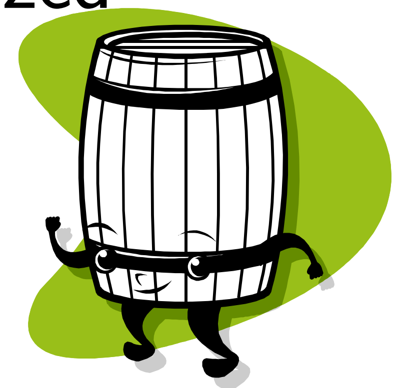
Integer multicommodity flow in a tree (MFT)

- **Input**: tree with edge capacities c_e; pairs of terminals, profit w_i for each commodity i.
  - Let *path(i)* denote path between terminals for i
- **Goal**: integers $x_i \geq 0$ such that for each e

$$\sum_{i:e\in path(i)} x_i \leq c_e$$

$$\text{such that} \quad \sum_i w_i x_i \quad \text{is maximized}$$

  - E.g. ship kegs on tree network
- APX-complete (GVY '93), 4-apx (CMS '03)



Key Notion: LP-Relative Approximation

- *Natural LP* for MFT: flow values *x* can be fractional, otherwise the same. LP-OPT can be found in poly-time. Note LP-OPT ≥ OPT.
- Def^n. An *LP-relative* α*–approx. algorithm* for a “max” problem always returns value at least LP-OPT/α; “min” is analogous
  - Same as “α–*approx.*” def^n with OPT ⇒ LP-OPT
  - Stronger notion
  - Abundant in papers but no common term?
    - E.g. 4-approx for MFT, min degree+1 spanning tree

Counting Lemma for MFT

- (KPP 08) "Let x be an extreme solution to the natural LP for MFT. If $x_i < 1$ for every commodity i, then some edge e^* has the following property:
  - At most 3 commodities i have $x_i > 0$ and i in *path*(e^*)"
- Same works for *covering version* of MFT
- ('09) Also holds for arc- and vertex-capacitated versions of MFT, with 3 replaced by 7
  - Won't state these versions explicitly from now on but all results in this talk go through

Counting Lemma Consequences

- i) Capacitated covering-MFT: some var has value ≥ 1/3 ⇒ 3-apx by iterated rounding

ratio violation

- ii) (1, +2)-approximation for MFT by iterated relaxation as mentioned earlier
- iii) (1, -2)-approximation for covering-MFT
- Key: these results are all LP-relative!

Results as Applied To MFT

- Let μ be minimum (edge/arc/v^x) capacity
- We get $1+O(1/\mu)$ approximation algorithm
  - Previous best ratio: constant (4/4/5)
  - Asymptotically optimal in terms of μ: $\exists \epsilon > 0 \forall \mu$, we get $1+O(\epsilon/\mu)$ inapproximability
  - Also get $1+O(1/\mu)$ approximation algorithm for covering-MFT

- Next up: sketches of the techniques

Mini-Technique: LP-Relative Covers Help Cover (e.g.: MFT)

- *There is a* (1+2/μ)*-apx alg for covering-MFT*
- Proof
  - Artificially increase all requirements c_e by 2
  - Scaling: if x is an optimum to the old LP, then x*(2+μ)/μ is a solution to the new LP
  - So LP-OPT' ≤ LP-OPT*(2+μ)/μ
  - Now apply the LP-relative (1, –2) approximation to the new requirements, gives a solution which meets old requirements and has cost at most

$$1 * \text{LP-OPT}' \leq \text{LP-OPT} * (2+\mu)/\mu$$

Technique: LP-Relative Covers Help Pack (e.g.: MFT)

- *There is a* $1+O(1/\mu)$-*apx alg for MFT*
- Proof
  - Let x be output of (1,+2) approx. algorithm
  - For each e let f_e be overload of e by x ($0 \leq f_e \leq 2$)
  - Look @ capacitated MFT-covering w/ requirements f and capacities x:
    - Scaling: $x*2/(2+\mu)$ is a feasible fractional covering
    - So LP-OPT' $\leq c(x)*2/(2+\mu)$
  - Run 3-apx alg for capacitated cover-MFT $\Rightarrow$ y
  - $x - y$ is a solution to original MFT instance and

$c(x-y) \geq c(x)*(1-3*2/(2+\mu)) \geq \text{LP-OPT}/(1+O(1/\mu))$

Generalized Results

- Consider a family of integer linear programs (A, b)
- Define $\mu := b_{min}$
- If "counting lemma" ⇒ known LP-rel consequences
  - (α', $-\nu'$) for uncapacitated covering min $\{cx \mid Ax \geq b, 0 \leq x\}$
  - (α, $+\nu$) for capacitated packing max $\{cx \mid Ax \leq b, 0 \leq x \leq d\}$
  - β for capacitated covering min $\{cx \mid Ax \geq b, 0 \leq x \leq d\}$
- Our techniques give in addition:
  - $\alpha'(1+\nu'/\mu) = \alpha'(1+O(1/\mu))$ for covering
  - $\alpha/(1-\beta\nu/(\mu+\nu)) =^* \alpha(1+O(1/\mu))$ for capacitated packing
  - *Remark*: can't hope to $1+O(1/\mu)$-approx *capacitated covering* in general since we can increase μ artificially

Related Open Problems

- Smallest *k*-edge-connected subgraph [GG 08]: best known apx ratio is $1 + 1/2k + O(1/k^2) + \ldots$
  - What about *min-cost k*-edge-connected multisubgraph?
    - Best known apx is 2, can we get $1+O(1/k)$?
- [*] Our packing techniques give no result when μ is too small. Is this avoidable or not?
  - E.g. *column-sparse packing ILPs* admit a counting lemma but need ad-hoc techniques to get const ratio for small μ
- Demand multicommodity flow in a tree?
  - Case of a star (*demand matching*) is well-studied and also a special case of column-sparse integer programs
  - $1+O(\text{dem}_{max}/\mu)^{1/2}$ best known, can we get $1+O(\text{dem}_{max}/\mu)$?

Merci!