On the fixed points of the iterated pseudopalindromic closure

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Outline

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Iterated palindromic closure

Palindromic closure

The *palindromic closure* of $w \in \mathcal{A}^*$, denoted w^+, is the shortest palindrome having w as prefix.

Iterated palindromic closure

Let $w \in \mathcal{A}^*$. The iterated palindromic closure of w, denoted $\mathrm{Pal}(w)$, is defined by

$$\mathrm{Pal}(\varepsilon) = \varepsilon, \quad \mathrm{Pal}(w) = \left(\mathrm{Pal}(w[1, n-1]) \cdot w[n]\right)^+.$$

Example

$\mathrm{Pal}(123) = (\mathrm{Pal}(12)3)^+ = ((\mathrm{Pal}(1)2)^+3)^+ = ((12)^+3)^+ = ((121)3)^+ = 1213121.$

Generalization of the iterated palindromic closure

It can naturally be generalized to an infinite word $u \in \mathcal{A}^\omega$ as

$$\mathrm{IPal}(u) = \lim_{n\to\infty} \mathrm{Pal}(u[1,n]).$$

Standard Sturmian and standard episturmian words

Let $w \in \mathcal{A}^\omega$. Then $\mathrm{IPal}(w)$ is a *standard episturmian word* if $|\mathcal{A}| \geq 3$ and if $|\mathcal{A}| = 2$ and $w \neq u\alpha^\omega$, it is a *standard Sturmian word*.

Example

Let consider the Tribonacci word $T = \mathrm{IPal}((123)^\omega)$. Then,

$$T = \underline{1}\underline{2}1\underline{3}121\underline{1}213121\underline{2}131211213121\cdots$$

We denote by $\Delta(T)$ the word that determines T, called the *directive word* of T. Here, $\Delta(T) = (123)^\omega$.

Pseudopalindrome

An *antimorphism* of $\mathcal{A}^*$ is a function $\theta : \mathcal{A}^* \to \mathcal{A}^*$ such that for all $u, v \in \mathcal{A}^*$, $\theta(uv) = \theta(v)\theta(u)$.

If $\theta^2 = \mathrm{id}$, then it is *involutive*.

θ-palindrome

For a fixed involutive antimorphism θ, a finite word $w \in \mathcal{A}^*$ is called a *θ-palindrome (pseudopalindrome)* if $\theta(w) = w$.

In the sequel, $R : \mathcal{A}^* \to \mathcal{A}^*$ is the involutive antimorphism defined as the reversal.

Example

For $w = abcab$, $R(w) = bacba$.

The R-palindromes are exactly the usual palindromes.

Involutive antimorphisms

Lemma

Let τ be an involutive permutation over the alphabet $\mathcal{A}$. Then $\theta = \tau \circ R = R \circ \tau$ is the unique involutive antimorphism on $\mathcal{A}^*$ that extends τ. Thus,

$$\theta(w) = \tau(w[n])\tau(w[n-1]) \cdots \tau(w[1]).$$

Any involutive antimorphism can be obtained that way.

Example

Let $\theta = R \circ \tau$, with $\tau(a) = b$ and $\tau(b) = a$. Then *abab* is a θ-palindrome. Indeed, *abab* is a fixed point under θ :

$$\theta(abab) = R(baba) = abab.$$

Iterated pseudopalindromic closure

The θ-*palindromic* (*pseudopalindromic*) *closure* of a word $w \in \mathcal{A}^+$, denoted $w^\oplus$, is the shortest θ-palindrome having w as prefix.

Example

Let θ be an involutive antimorphism such that $\tau(1) = 3$, $\tau(2) = 4$, $\tau(3) = 1$ and $\tau(4) = 2$, and let $w = 123124$. Then

$$w^\oplus = 1231 \cdot 24 \cdot \theta(1231) = 1231 \cdot 24 \cdot 3143.$$

The *iterated pseudopalindromic closure*, denoted Pal_θ, is naturally defined by $\mathrm{Pal}_\theta(\varepsilon) = \varepsilon$, and for $w \in \mathcal{A}^*$,

$$\mathrm{Pal}_\theta(w) = \left(\mathrm{Pal}_\theta(w[1, n-1])w[n]\right)^\oplus.$$

And $\mathrm{IPal}_\theta = \lim_{n\to\infty} \mathrm{Pal}_\theta(w[1, n])$.

Questions

1. What do the fixed points of the iterated pseudopalindromic closure look like ?
2. How many are they ?
3. Do they have remarkable combinatorial properties ?

Existence (1/2)

Example

$$\begin{aligned}
\mathrm{IPal}_R(abx\cdots) &= \underline{ab}a\underline{x}\cdots \\
\mathrm{IPal}_R^2(abx\cdots) &= \underline{ab}a\underline{a}ba\underline{x}\cdots \\
\mathrm{IPal}_R^3(abx\cdots) &= \underline{ab}a\underline{a}ba\underline{a}ba\underline{b}aabaaba\underline{a}babaabaaba\underline{x}\cdots .
\end{aligned}$$

Let E be the involutive antimorphism defined by $\mathsf{E} = R \circ \tau$, with $\tau(a) = b$ and $\tau(b) = a$. Then

$$\begin{aligned}
\mathrm{IPal}_{\mathsf{E}}(abx\cdots) &= \underline{a}b\underline{b}aab\underline{x}\cdots \\
\mathrm{IPal}_{\mathsf{E}}^2(abx\cdots) &= \underline{a}b\underline{b}aab\underline{b}aab\underline{a}bbaabbaab\underline{a}bbaabbaab\underline{b}aababbaabbaababbaabbaab\underline{x}\cdots
\end{aligned}$$

Existence (2/2)

Theorem and definition

Over a k-letter alphabet, with $k \geq 2$, there are 3 kinds of fixed points having at least 2 different letters, only depending on the first letters of the word and the involutory antimorphism $\theta = R \circ \tau$ considered.

1. When $\tau(a) = a$ and $\tau(b) = b$, with $a \neq b$, for a fixed $n \geq 1$, IPal_θ has a unique fixed point beginning with $a^n b$, denoted $\mathbf{s}_{R,n,a,b}$, which equals
$$\mathbf{s}_{R,n,a,b} = \lim_{i \to \infty} \mathrm{Pal}^i(a^n b) = \underline{a}^n \underline{b} a^n (\underline{a} b a^n)^{n+1} \underline{b} (a^{n+1} b)^{n+1} a^n \underline{a} \cdots .$$
2. When $\tau(a) = a$ and $\tau(b) = c$ for pairwise different letters a, b, c, for a fixed $n \geq 1$, IPal_θ has a unique fixed point beginning with $a^n b$, denoted by $\mathbf{s}_{\mathcal{H},n,a,b,c}$, which equals
$$\mathbf{s}_{\mathcal{H},n,a,b,c} = \lim_{i \to \infty} \mathrm{Pal}^i_{\mathcal{H}}(a^n b) = \underline{a}^n \underline{b} c a^n \underline{c} b a^n b c a^n (\underline{a} b c a^n c b a^n b c a^n)^n \underline{c} \cdots .$$
3. When $\tau(a) = b$ and $\tau(b) = a$, with $a \neq b$, IPal_θ has a fixed point beginning with $a^n b$ only if $n = 1$. It is denoted by $\mathbf{s}_{\mathrm{E},a,b}$ and equals
$$\mathbf{s}_{\mathrm{E},a,b} = \lim_{i \to \infty} \mathrm{Pal}^i_{\mathrm{E}}(a) = \underline{a} b \underline{b} a a b \underline{b} a a b \underline{a} b b a a b b a a b \underline{a} b b a a b b a a b \underline{b} \cdots .$$

Combinatorial properties of $\mathbf{s}_{\mathbf{R},\mathbf{n},\mathbf{a},\mathbf{b}}$

1. For a fixed positive $n \in \mathbb{N}$, $\mathbf{s}_{R,n,a,b}$ is not ultimately periodic and consequently, is standard Sturmian (using Droubay, Justin, Pirillo - 2001).
2. For a fixed $n \in \mathbb{N}$, $\mathbf{s}_{R,n,a,b}$ is not a fixed point of a nontrivial morphism (using Arnoux, Rauzy - 1991 and Crisp et al - 1993).
3. $\mathbf{s}_{R,n,a,b}$ is $(n+4)$-th power-free, but contains $(n+3)$-th powers (using Vandeth - 2000).
4. For any $n \geq 1$, $\alpha_{n,a,b}$ is transcendental (using Adamczewski, Bugeaud - 2007).

Combinatorial properties of $\mathbf{s}_{\mathsf{E},\mathbf{a},\mathbf{b}}$ (1/2)

Useful property [de Luca, De Luca - 2006]

Let $\theta = \tau \circ R$ be an involutory antimorphism over an alphabet $\mathcal{A}$, with μ_θ the morphism defined for all a in $\mathcal{A}$, by $\mu(a) = a$ if $a = \tau(a)$ and by $\mu(a) = a\tau(a)$ otherwise.

Then, for any $\mathbf{w} \in \mathcal{A}^\omega$ and for any involutory antimorphism θ, one has

$$\mathrm{IPal}_\theta(\mathbf{w}) = \mu_\theta(\mathrm{IPal}(\mathbf{w})).$$

Idea

First consider $\mathbf{w}_{\mathsf{E}} = \mathrm{IPal}(\mathbf{s}_{\mathsf{E},\mathbf{a},\mathbf{b}})$ and then, extend its properties to $\mu_{\mathsf{E}}(\mathrm{IPal}(\mathbf{s}_{\mathsf{E},\mathbf{a},\mathbf{b}})) = \mathbf{s}_{\mathsf{E},\mathbf{a},\mathbf{b}}$.

$$\mathbf{w}_{\mathsf{E}} = \underline{ab}a\underline{b}a\underline{a}baba\underline{a}baba\underline{b}aababaababa\underline{b}aababaababa\cdots$$

Combinatorial properties of $\mathbf{s}_{\mathrm{E},\mathbf{a},\mathbf{b}}$ (2/2)

Property of $\mathbf{w}_{\mathrm{E}}$

$\mathbf{w}_{\mathrm{E}}$ is not ultimately periodic, and consequently, is a Sturmian word.

Lemma

An infinite word $\mathbf{w}$ over $\mathcal{A}$ is ultimately periodic if and only if $\mu_\theta(\mathbf{w})$ is so.

$\Longrightarrow$ $\mathbf{s}_{\mathrm{E},\mathbf{a},\mathbf{b}}$ is not ultimately periodic.

Lemma

$\mathbf{w}_{\mathrm{E}}$ is not a fixed point for some non-trivial morphism $\Longrightarrow$ $\mathbf{s}_{\mathrm{E},a,b}$ is not a fixed point for some non-trivial morphism.

Proposition

$\mathbf{w}_{\mathrm{E}}$ and $\mathbf{s}_{\mathrm{E},\mathbf{a},\mathbf{b}}$ both contain 4-th powers, but no 5-th power words (using Shur - 2000).

Combinatorial properties of $\mathbf{s}_{\mathcal{H},\mathbf{n},\mathbf{a},\mathbf{b},\mathbf{c}}$ (1/3)

Recall : $\tau(a) = a$, $\tau(b) = c$ and $\tau(c) = b$, and $\theta = R \circ \tau$.

First consider $\mathbf{w}_{\mathcal{H}} = \mathrm{IPal}(\mathbf{s}_{\mathcal{H},\mathbf{n},\mathbf{a},\mathbf{b},\mathbf{c}})$ and then extend its properties to $\mu_{\mathcal{H}}(\mathrm{IPal}(\mathbf{s}_{\mathcal{H},\mathbf{n},\mathbf{a},\mathbf{b},\mathbf{c}})) = \mathbf{s}_{\mathcal{H},\mathbf{n},\mathbf{a},\mathbf{b},\mathbf{c}}$.

$$\mathbf{w}_{\mathcal{H},\mathbf{n}} = \mathrm{IPal}(\mathbf{s}_{\mathcal{H},n,a,b,c}) = \underline{a}^n \underline{b} a^n \underline{c} a^n b a^n \underline{a} b a^n c a^n b a^n \cdots$$

Property of $\mathbf{w}_{\mathcal{H},\mathbf{n}}$

$\mathbf{w}_{\mathcal{H},\mathbf{n}}$ is not ultimately periodic, and consequently, is a strict standard episturmian word.

$\Longrightarrow$ $\mathbf{s}_{\mathcal{H},\mathbf{n},\mathbf{a},\mathbf{b},\mathbf{c}}$ is not ultimately periodic.

Proposition

$\mathbf{w}_{\mathcal{H},\mathbf{n}}$ is not a fixed point of a nontrivial morphism (using Justin, Pirillo - 2002).

Combinatorial properties of $\mathbf{s}_{\mathcal{H},\mathbf{n},\mathbf{a},\mathbf{b},\mathbf{c}}$ (2/3)

Generalization of a result of Justin, Pirillo - 2002 for strict episturmian word having periodic directive word :

Proposition

Let $\mathbf{s}$ be a strict standard episturmian word directed by a word Δ and let ℓ denotes the greatest integer such that α^ℓ is a factor of Δ with α a letter. Assume Δ contains at least one factor $aua^\ell va$ with a a letter and u, v non empty words that do not contain the letter a. Then $\mathbf{s}$ is $(\ell + 3)$-th power-free but contains an $(\ell + 2)$-th power.

Corollary

The words $\mathbf{w}_{\mathcal{H},\mathbf{n}}$ are $(n + 4)$-th power free but contain $(n + 3)$-th powers.

Combinatorial properties of $\mathbf{s}_{\mathcal{H},\mathbf{n},\mathbf{a},\mathbf{b},\mathbf{c}}$ (3/3)

Let $\mathbf{s}_{\mathcal{H},n,a,b,c}$ be a fixed point of the $\mathrm{IPal}_{\mathcal{H}}$ operator, for a fixed n. Then $\mathbf{s}_{\mathcal{H},n,a,b,c}$ satisfies the following properties :

1. It is not an episturmian word, but is a pseudostandard word.
2. It is not a fixed point for some non trivial morphism.
3. It is $(n+4)$-th power-free but contains $(n+3)$-th powers.
4. The frequencies of the letters b and c are equal.

Open problems

- What is the critical exponent : its value, the number of occurences, their positions, etc.
- Does there exist a connection between $s_{R,n,a,b}$ and $s_{R,n+1,a,b}$? between $s_{\mathcal{H},n,a,b,c}$ and $s_{\mathcal{H},n+1,a,b,c}$?
- Can we give a geometric interpretation of the iterated palindromic (pseudopalindromic) closure ?
- What are the letter frequencies of $s_{\mathcal{H},n,a,b,c}$?
- Any remarkable combinatorial properties ?