Switching operations and binary codes of Hadamard matrices

CanaDAM 2009, Montréal

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28 May 2009
Hadamard matrices

- matrix elements 1, −1 (written as ‘−’)
- orthogonal rows
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- orthogonal rows

Negations and permutations of rows and columns produce an equivalent Hadamard matrix:

\[
\begin{bmatrix}
-1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1 \\
\end{bmatrix} \sim
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 \\
\end{bmatrix}
\]
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\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 \\
\end{bmatrix}
$$

The matrix on the right is normalized.
Size $n$ of Hadamard matrix must be 1, 2, or multiple of 4.
Existence of Hadamard matrices

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Question of existence for every $n = 4k$ is still unresolved...
Existence of Hadamard matrices

Size $n$ of Hadamard matrix must be 1, 2, or multiple of 4.

Question of existence for every $n = 4k$ is still unresolved. . .

. . . but astronomical growth in the number of equivalence classes seems to occur.
### Numbers of equivalence classes

<table>
<thead>
<tr>
<th>$n$</th>
<th>num. equiv. classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>1 Hadamard</td>
</tr>
<tr>
<td>16</td>
<td>5 Todd, M. Hall</td>
</tr>
<tr>
<td>20</td>
<td>3 M. Hall</td>
</tr>
<tr>
<td>24</td>
<td>60 Ito, Leon, Longyear; Kimura</td>
</tr>
<tr>
<td>28</td>
<td>487 Kimura</td>
</tr>
<tr>
<td>32</td>
<td>$\geq 13,561,612$ O.</td>
</tr>
<tr>
<td>36</td>
<td>$\geq 18,000,000$ O.</td>
</tr>
<tr>
<td>40</td>
<td>$\geq 3.66 \times 10^{11}$ Lam, Lam, Tonchev</td>
</tr>
</tbody>
</table>
Goal of this talk:

To propose some ideas for understanding and managing the combinatorial explosion using codes and switching operations.
If $H$ is an $n \times n$ Hadamard matrix, $n > 2$, then rows 1–3 of $H$ may be put in the form

$$
\begin{pmatrix}
\frac{n}{4} & \frac{n}{4} & \frac{n}{4} & \frac{n}{4} \\
1 \ldots 1 & 1 \ldots 1 & 1 \ldots 1 & 1 \ldots 1 \\
1 \ldots 1 & 1 \ldots 1 & \ldots \ldots & \ldots \ldots \\
1 \ldots 1 & \ldots \ldots & 1 \ldots 1 & \ldots \ldots
\end{pmatrix}.
$$
3-normalization

If $H$ is an $n \times n$ Hadamard matrix, $n > 2$, then rows 1–3 of $H$ may be put in the form

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\begin{pmatrix}
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1 \ldots 1 & 1 \ldots 1 & 1 \ldots 1 & 1 \ldots 1 \\
1 \ldots 1 & 1 \ldots 1 & \ldots \ldots & \ldots \ldots \\
1 \ldots 1 & \ldots \ldots & 1 \ldots 1 & \ldots \ldots \\
\end{pmatrix}
$$

$$
\implies 4 \mid n
$$
The fourth row is not uniquely determined.

\[
\begin{bmatrix}
\frac{n}{4} & \frac{n}{4} & \frac{n}{4} & \frac{n}{4} \\
1 & 1 & 1 & -
\end{bmatrix}
\]

Orthogonality $\Rightarrow$ vectors $a, b, c, d$ have same number of 1s.

Switching operations and binary codes

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\[
\begin{pmatrix}
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1 \ldots 1 & 1 \ldots 1 & 1 \ldots 1 & 1 \ldots 1 \\
1 \ldots 1 & 1 \ldots 1 & \ldots & \ldots \\
1 \ldots 1 & \ldots & 1 \ldots 1 & \ldots \\
a & -b & -c & d
\end{pmatrix}
\]

Orthogonality \Rightarrow vectors \(a, b, c, d\) have same number of 1s.
Type-\(j\) quadruples

Let \(\#(r)\) denote numbers of 1s in vector \(r\).

A set of four rows is a type-\(j\) quadruple if

1. \(\#(a) = j\), or
2. \(\#(a) = n/4 - j\).
Type-\(j\) quadruples

Let \(\#(r)\) denote numbers of 1s in vector \(r\). A set of four rows is a type-\(j\) quadruple if
- \(\#(a) = j\), or
- \(\#(a) = n/4 - j\).

Type-0 quadruples can exist only when \(n = 4\) or \(8 \mid n\). (Orthogonality of rows 4 and 5 fails, otherwise.)
Switching type-0 quadruples

Consider two forms of type-0 quadruple,

\[
Q = \begin{pmatrix}
1 \ldots 1 & 1 \ldots 1 & 1 \ldots 1 & 1 \ldots 1 \\
1 \ldots 1 & 1 \ldots 1 & \cdots & \cdots \\
1 \ldots 1 & \cdots & 1 \ldots 1 & \cdots \\
1 \ldots 1 & \cdots & \cdots & 1 \ldots 1
\end{pmatrix}
\]

\[
Q' = \begin{pmatrix}
\cdots & 1 \ldots 1 & 1 \ldots 1 & 1 \ldots 1 \\
\cdots & 1 \ldots 1 & \cdots & \cdots \\
\cdots & \cdots & 1 \ldots 1 & \cdots \\
\cdots & \cdots & \cdots & 1 \ldots 1
\end{pmatrix}
\]

Then

\[
H = \begin{pmatrix}
Q \\
A
\end{pmatrix}
\]

is Hadamard \iff

\[
H' = \begin{pmatrix}
Q' \\
A
\end{pmatrix}
\]

is Hadamard
Switching type-0 quadruples

Consider two forms of type-0 quadruple,

\[ Q = \begin{pmatrix} 1 \ldots 1 & 1 \ldots 1 & 1 \ldots 1 & 1 \ldots 1 \\ 1 \ldots 1 & 1 \ldots 1 & \cdot \cdot \cdot & \cdot \cdot \cdot \\ 1 \ldots 1 & \cdot \cdot \cdot & 1 \ldots 1 & \cdot \cdot \cdot \\ 1 \ldots 1 & \cdot \cdot \cdot & \cdot \cdot \cdot & 1 \ldots 1 \\ \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot \end{pmatrix} \]

\[ Q' = \begin{pmatrix} 1 \ldots 1 & 1 \ldots 1 & 1 \ldots 1 \\ 1 \ldots 1 & 1 \ldots 1 & \cdot \cdot \cdot \\ 1 \ldots 1 & \cdot \cdot \cdot & 1 \ldots 1 \\ 1 \ldots 1 & \cdot \cdot \cdot & \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot \end{pmatrix} \]

Then

\[ H = \begin{pmatrix} Q \\ A \end{pmatrix} \text{ is Hadamard} \iff H' = \begin{pmatrix} Q' \\ A \end{pmatrix} \text{ is Hadamard} \]

\[ H \text{ and } H' \text{ are generally not equivalent.} \]
Switching produces many new Hadamard matrices

\[ n = 16 : \]
From any equivalence class, all five classes can be obtained by a sequence of switching operations.
Switching produces many new Hadamard matrices

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From any equivalence class, all five classes can be obtained by a sequence of switching operations.

\[ n = 32 : \]
From the Sylvester equivalence class one obtains exactly 13,561,612 classes by switching. Additional classes are known, so this is only a lower bound.
Switching produces many new Hadamard matrices

\( n = 24 \):

The 60 equivalence classes fall into nine switching classes of sizes

<table>
<thead>
<tr>
<th>( j )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>num.</td>
<td>8</td>
<td>8</td>
<td>17</td>
<td>10</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>60</td>
</tr>
</tbody>
</table>

As representatives of the nine classes we may take the eight matrices of the form

\[
\begin{bmatrix}
H_1 & H_1 \\
H_2 & -H^2
\end{bmatrix},
\]

with \( H_1 \) and \( H_2 \) Hadamard matrices of size 12, and the Paley matrix.
A linear code is a vector space over GF(2). Codes are formed from incidence structures by taking the span of the rows of the incidence matrix.
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To obtain a \( \{0, 1\} \) matrix from \( H \), normalize \( H \); replace \(-1\) with \(0\). Call the resulting matrix \( A \).
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To obtain a \( \{0, 1\} \) matrix from \( H \), normalize \( H \); replace \(-1\) with \(0\). Call the resulting matrix \( A \).

Two codes: the row code \( C(H) \) and the column code \( C(H^T) \).
Properties of the code

- Every codeword is orthogonal to $j = 111 \ldots 1$. 

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If $n \equiv 0 \pmod{8}$ then $C(H) \subseteq C(H)^\perp$ (it’s self-orthogonal) $\implies \dim C(H) \leq n/2$. (Follows from 3-normalization.)
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- If \( n \equiv 0 \pmod{8} \) then \( C(H) \subseteq C(H)^\perp \) (it’s self-orthogonal) \( \implies \dim C(H) \leq n/2 \). (Follows from 3-normalization.)
- If \( n \equiv 8 \pmod{16} \) then \( C(H) = C(H)^\perp \) (it’s self-dual) \( \implies \dim C(H) = n/2 \). (From properties of invt. factors of \( H \).)
Summary of codes, $n \equiv 0 \pmod{8}$

$C(H)$ and $C(H^T)$ are
- doubly-even,
- self-orthogonal and possibly self-dual,
- always self-dual when $n \equiv 8 \pmod{16}$,
- of dimension at most $n/2$,
- of dimension at least $\lceil \log_2 n \rceil + 1$ (W.D. Wallis).
$n \equiv 0 \pmod{8}$: weight-4 codewords

Weight-4 codewords have special significance. Why?
\[ n \equiv 0 \pmod{8}: \text{weight-4 codewords} \]

Weight-4 codewords have special significance. Why?

\[
\begin{pmatrix}
A \\
\vdots \\
0
\end{pmatrix}
\begin{pmatrix}
i \\
j \\
k \\
l
\end{pmatrix}
\begin{pmatrix}
1 \ldots 0 \ldots 1 \ldots 0 \ldots 1 \ldots 0 \\
1 \ldots 0 \ldots 1 \ldots 0 \ldots 0 \ldots 1 \\
1 \ldots 0 \ldots 0 \ldots 1 \ldots 1 \ldots 0 \\
1 \ldots 0 \ldots 0 \ldots 1 \ldots 0 \ldots 1
\end{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
1 \\
1 \\
1
\end{pmatrix}
\]
Consider the column code, $C(H^T)$; regard codewords as column vectors.
Let $w \in C(H^T)$ be a weight-4 codeword with support in rows $i, j, k, l$. 

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Let $w \in C(H^T)$ be a weight-4 codeword with support in rows $i, j, k, l$.

$$\implies \{i, j, k, l\} \text{ is a type-0 quadruple.}$$

(Follows from self-orthogonality of $C(H^T)$ and orthogonality of $w$ with every column of $A$.)
$n \equiv 0 \pmod{8}$: weight-4 codewords

**Partial converse:**
Suppose that rows $i, j, k, l$ form a type-0 quadruple of $H$. Then $w \in C(H^T)$. If $C(H^T)$ is self-dual, then $w$ is in the code; otherwise it may or may not be. Why?
The columns of $A$ generate $C(H^T)$ and contain an even number of 1s in rows $i, j, k, l$. Hence $w$ is orthogonal to every codeword in $C(H^T)$. 

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$n \equiv 0 \pmod{8}$: weight-4 codewords

**Partial converse:**
Suppose that rows $i, j, k, l$ form a type-0 quadruple of $H$.

Then $w \in \mathcal{C}(H^T)^\perp$.
If $\mathcal{C}(H^T)$ is self-dual, then $w$ is in the code; otherwise it may or may not be.
Partial converse:
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Why?

The columns of \(A\) generate \(C(H^T)\) and contain an even numbers of 1s in rows \(i, j, k, l\). Hence \(w\) is orthogonal to every codeword in \(C(H^T)\).
$n \equiv 0 \pmod{8}$: weight-4 codewords.

Summary

If $n \equiv 8 \pmod{16}$ then $C(H^T) = C(H^T)^\perp$. Hence there is a one-to-one correspondence between type-0 quadruples in $H$ and weight-4 codewords in $C(H^T)$. 
If $n \equiv 8 \pmod{16}$ then $C(H^T) = C(H^T) \perp$. Hence there is a one-to-one correspondence between type-0 quadruples in $H$ and weight-4 codewords in $C(H^T)$.

If $n \equiv 0 \pmod{16}$ then $C(H^T) \subseteq C(H^T) \perp$. In this case we must distinguish between type-0 quadruples that correspond to weight-4 codewords in $C(H^T)$ and those that do not.
The code of the Sylvester matrix is the Reed-Müller code. For \( n \geq 16 \) it contains no weight-4 codewords, but Sylvester matrices contain numerous type-0 quadruples.
$n \equiv 0 \pmod{16}$: extremal examples

The code of the Sylvester matrix is the Reed-Müller code. For $n \geq 16$ it contains no weight-4 codewords, but Sylvester matrices contain numerous type-0 quadruples.

Opposite extreme: matrices with self-dual codes. (There are two of size 16.)
Switching and codes

Let \( \{i, j, k, l\} \) be a type-0 quadruple of \( H \), and let \( w \) be the weight-4 word with support in rows \( i, j, k, l \). Let \( H' \) be the Hadamard matrix that results from switching. Switching can be regarded as adding \( w \) to certain columns of the generator matrix \( A \) to produce a new generator matrix \( A' \). Consequently

\[
\langle C(H^T), w \rangle = \langle C(H'^T), w \rangle
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\[
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\]

Three cases:

1. \( w \in C(H^T), \ w \in C(H'^T) \implies C(H^T) = C(H'^T) \).
2. \( w \in C(H^T), \ w \notin C(H'^T) \implies C(H'^T) \subset \langle C(H'^T), w \rangle = C(H^T); \dim C(H^T) = 1 + \dim C(H'^T). \) (Or the reverse: \( H \leftrightarrow H' \).)
3. \( w \notin C(H^T), \ w \notin C(H'^T) \implies \dim C(H^T) = \dim C(H'^T). \)
Case (1) holds.

$C(H^T)$ and $C(H'^T)$ are the same self-dual code, which contains $w$. 

$n \equiv 8 \pmod{16}$

Case (1) holds. \( C(H^\top) \) and \( C(H'^\top) \) are the same self-dual code, which contains \( w \).

Conjecture: This is the only situation in which Case (1) holds.
Conjecture: Either Case (2) or Case (3) holds.
Switching and codes: $n \equiv 0 \pmod{16}$

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More specific conjecture:
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- When \( n \equiv 16 \pmod{32} \) Case (2) holds. Exactly one of \( C(H^T) \) and \( C(H'^T) \) contains \( w \).
Conjecture: Either Case (2) or Case (3) holds.

More specific conjecture:

- When $n \equiv 16 \pmod{32}$ Case (2) holds. Exactly one of $C(H^T)$ and $C(H'^T)$ contains $w$.

- When $n \equiv 0 \pmod{32}$ either Case (2) or Case (3) holds. (It is possible that neither $C(H^T)$ nor $C(H'^T)$ contains $w$.)

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Further consequences:

- $C(H^T)$ is invariant under switching.
- The set of type-0 quadruples in $H$ is invariant under switching.
- Switching operations can be composed, and therefore form a group, the *switching group*. 
$n \equiv 0 \pmod{8}$: weight-4 codewords

Define $\mathcal{F}(H^T) \subseteq \mathcal{C}(H^T)$ to be the subcode generated by the weight-4 codewords of $\mathcal{C}(H^T)$.
What are the possible structures of $\mathcal{F}(H^T)$?
Define $\mathcal{F}(H^T) \subseteq \mathcal{C}(H^T)$ to be the subcode generated by the weight-4 codewords of $\mathcal{C}(H^T)$.

What are the possible structures of $\mathcal{F}(H^T)$?

Conway and Pless: Any doubly-even, self-orthogonal code generated by weight-4 codewords is a direct sum of three types of code:

- $d_{2k}$, $k \geq 2$,
- $e_7$, the even-weight subcode of the $[7, 4, 3]$ Hamming code,
- $e_8$, the $[8, 4, 4]$ extended Hamming code.
The code $d_{2k}$...

... has length $2k$ and is generated by the $k - 1$ row vectors

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots \\
1 & 1 & 0 & 0 & 0 & 0 & \ldots & 1 & 1
\end{bmatrix}$$

It contains $\binom{k}{2}$ codewords of weight 4.
The code $e_7$...

... has length 7 and is generated by the three row vectors

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
\]

It contains seven codewords of weight 4.
The code $e_8$ has length 8 and is generated by the four row vectors:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}
\]

It contains 14 codewords of weight 4.
Restrictions on $F(H^T)$

- The code of $H_8$ is $e_8$; $e_8$ does not appear in the code of any $H_n$ for any other $n \equiv 8 \pmod{16}$.
- $e_7$ does not appear in the code of any $H_n$ for $n = 24$ (Assmus and Key) or $n = 40$. 
$n \equiv 8 \pmod{16}$: The switching group

The switching group is a subgroup acting on a given Hadamard matrix is a subgroup of the orthogonal group. It is the direct sum of the switching groups acting on the components $d_{2k}$ and $e_7$ of which $\mathcal{F}(H^\top)$ is composed.

On $d_4$, the group is a 2-element group. On $d_{2k}$, $k \geq 3$ the group has $2^{k-1}k!$ elements. The center is the set of elements that permute the rows of an even number of duads of $d_{2k}$. Since row permutation does not change the Hadamard equivalence class, we take the quotient by this center to obtain the group of moves that potentially do change the Hadamard equivalence class. This group is isomorphic to $S_k$. 
Example: \( n = 24 \)

Since \( 24 \equiv 8 \pmod{16} \),
- the code is self-dual,
- there is a one-to-one correspondence between weight-4 codewords and type-0 quadruples,
- switching does not change the code,
- the subcode \( \mathcal{F}(H^T) \) must be a direct sum of codes \( d_{2k} \) and \( e_7 \) only; \( e_7 \) cannot occur in size 24.
There are nine self-dual, doubly-even codes of length 24 (Conway and Pless). Of these, three contain either $e_7$ or $e_8$ and are therefore not relevant to Hadamard matrices. The remaining six codes include the extended Golay code and five codes with $F = d_{24}, 2d_{12}, 3d_{8}, 4d_{6}, 6d_{4}$. All six occur. (Assmus and Key)
Example: \( n = 24 \)

The code of \( H = \begin{bmatrix} H_1 & H_1 \\ H_2 & -H^2 \end{bmatrix} \) where \( H_1 \) and \( H_2 \) are \( 12 \times 12 \) Hadamard matrices is the code containing \( d_{24} \). The code of \( H^T \) may be any of the six codes. There are eight such transposed matrices \( H \) up to equivalence. The codes containing \( 4d_6 \) and \( 6d_4 \) each occur twice; the other four codes occur once. We have eight switching classes of sizes \( 8 \ (d_{24}), 8 \ (2d_{12}), 17 \ (3d_8), 10 \) and \( 5 \ (4d_6), 5 \) and \( 5 \ (6d_4) \), and \( 1 \) (the extended Golay code which, of course, has no weight-4 codewords).

The code of the Paley matrix is also the extended Golay code.
Example: $n = 16$

- Must consider non-self-dual codes as well.
- Switching changes the dimension of the code by $\pm 1$.
- The smaller code is a subcode of the larger.
- Every self-orthogonal, doubly even code is a subcode of a self-dual, doubly-even code.
- The latter have been classified up to $n = 32$ by Conway and Pless. For $n = 16$ there are two self-dual codes, $e_8 \oplus e_8$ and $\langle d_{16} \cup \{(1010 \ldots 10)\}\rangle$.
- Each of the self-dual codes corresponds to one Hadamard matrix.
Example: \( n = 16 \)

Mike Grilli (Research Experience for Undergraduates, Indiana University 2008) worked out a method for systematically identifying subcodes of self-dual codes that are potentially relevant to Hadamard matrices. For length 16, the subcodes of \( e_8 \oplus e_8 \) and \( \langle d_{16} \cup \{(1010 \ldots 10)\} \rangle \) include

- one relevant subcode of dimension 7, which is common to the two self dual codes, and contains \( d_8 \oplus d_8 \),
- one relevant subcode of dimension 6, which contains \( d_4 \oplus d_4 \oplus d_4 \oplus d_4 \),
- one relevant subcode of dimension 5, the Reed-Müller code, which has no weight-4 codewords.