Variations on Pebbling and Graham's Conjecture

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Talk structure

Pebbling numbers

Products and Graham's Conjecture

Variations

- Optimal pebbling
- Weighted graphs
- Choosing target distributions

Basic notions

Distributions on G:
  D:V(G)→ℕ

D(v) counts pebbles on v

Pebbling moves

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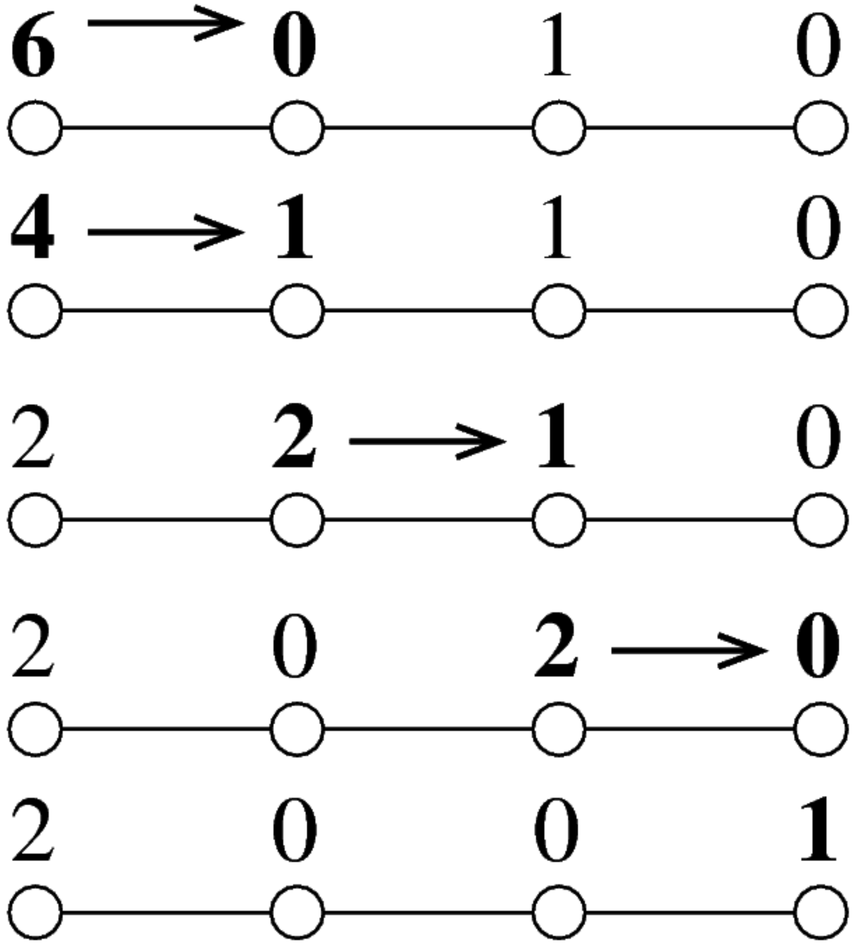
Pebbling moves



Pebbling numbers

π(G, D) is the number of pebbles required to ensure that D can be reached from any distribution of π(G, D) pebbles.

If S is a ***set*** of distributions on G

$$\pi(G,S) = max_{D \in S} \pi(G,D)$$

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Optimal pebbling number:

π*(G, S) is the number of pebbles required in ***some*** distribution from which every D∈S can be reached

Common pebbling numbers

$S_1(G)$: 1 pebble anywhere

$$\pi(G)=\pi(G,S_1(G))$$

(*pebbling number*)

$S_t(G)$: t pebbles on some vertex

$$\pi_t(G)=\pi(G,S_t(G))$$

(*t-pebbling number*)

δ_v: One pebble on v

$$\pi(G,v)=\pi(G,\delta_v)$$

S(G, t) and π(G, t)

S(G, t): all distributions with a total of t pebbles (anywhere on the graph)

Conjecture

$$\pi(G, S(G, t)) = \pi(G, S_t(G))$$

i.e. hardest-to-reach ***target*** configurations have all pebbles on one vertex

True for K_n, C_n, trees

Cover pebbling

Γ(G): one pebble on every vertex

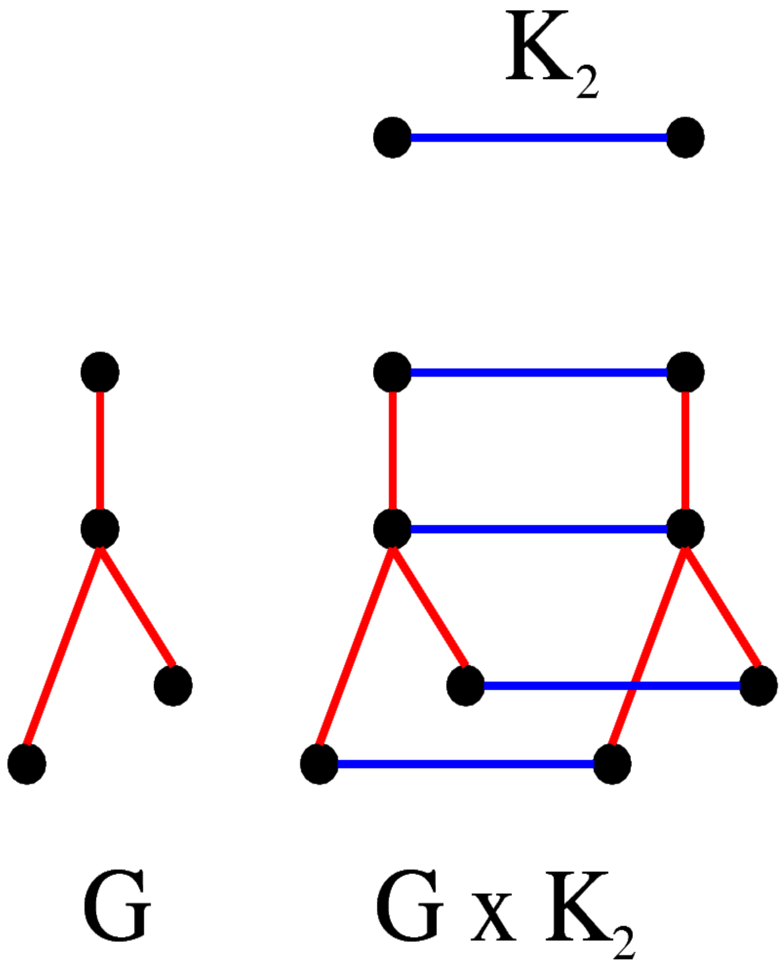
$$\gamma(G)=\pi(G,\Gamma(G))$$

(*cover pebbling number*)

Sjöstrund: If D(v) ≥ 1 for all vertices v, then there is a critical distribution with all pebbles on one vertex.

(A *critical* distribution has one pebble less than the required number and cannot reach some target distribution)

Cartesian products of graphs



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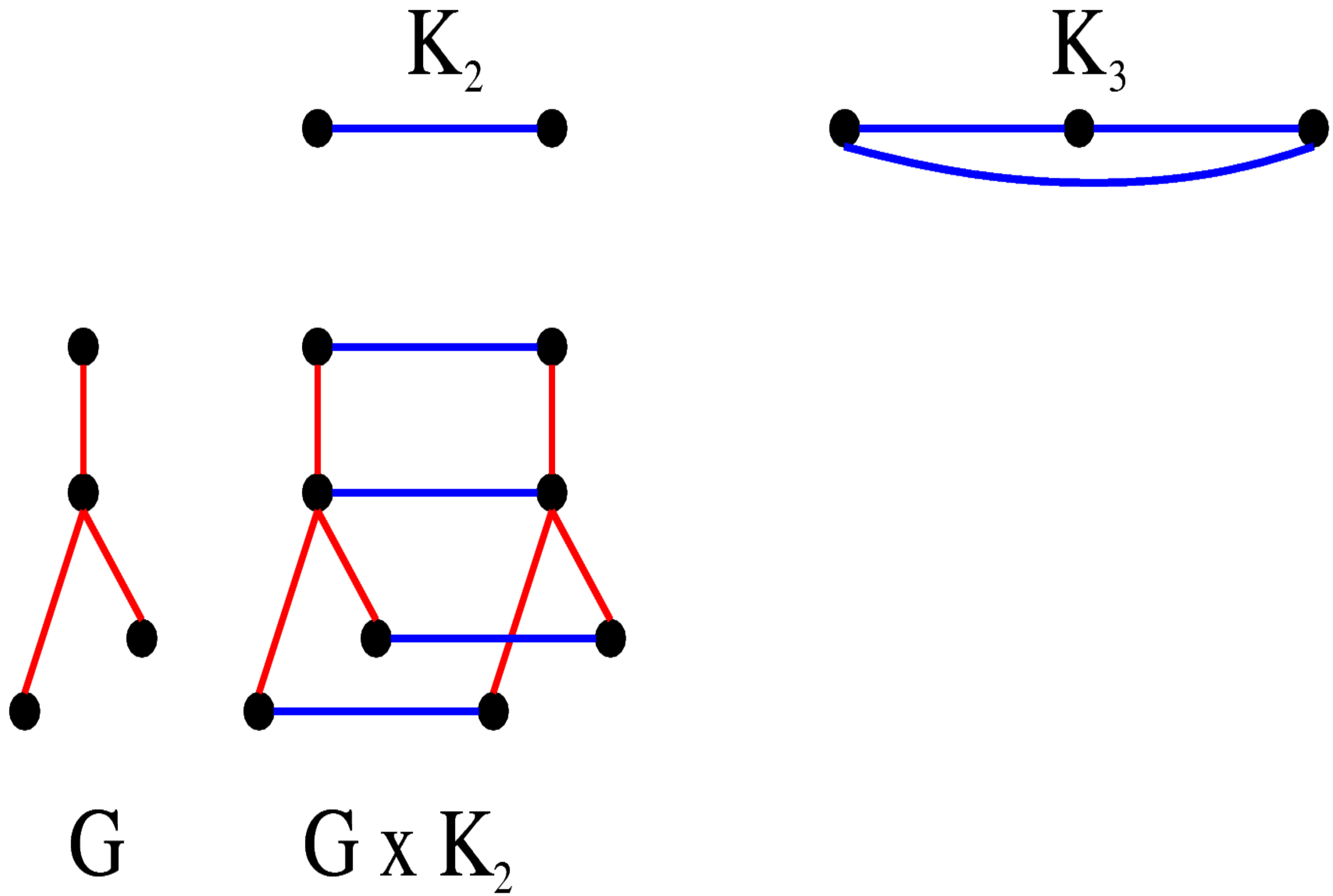

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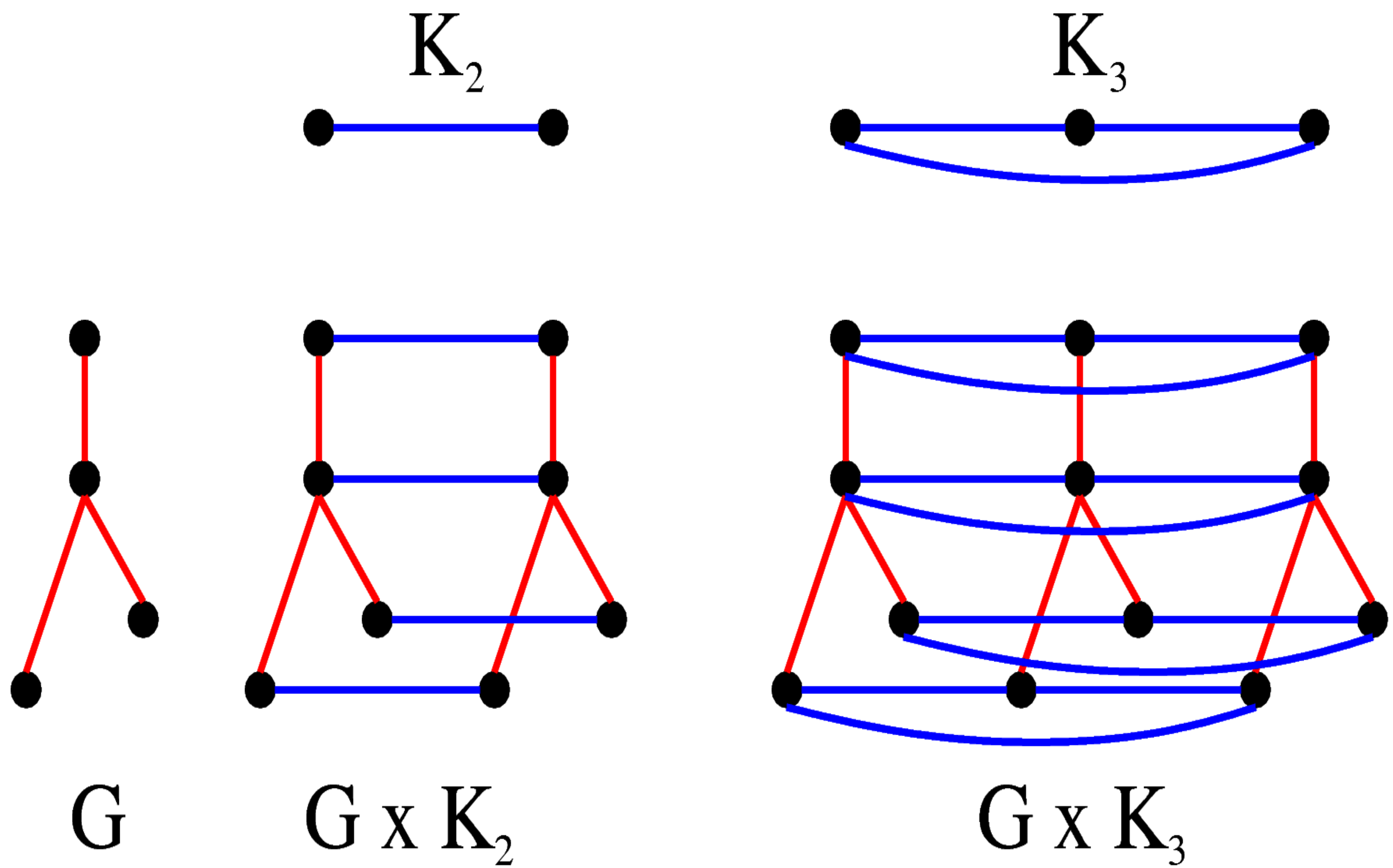


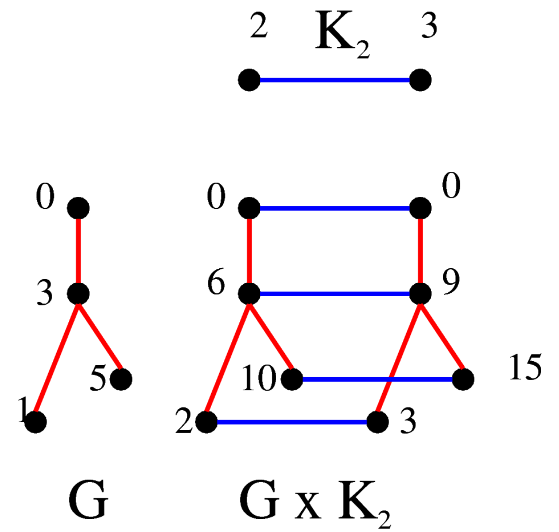
Products of distributions

Product of distributions:

D_1 on G; D_2 on H

then $D_1 \cdot D_2$ on GxH

$$D_1 \cdot D_2((v, w)) = D_1(v)\, D_2(w)$$



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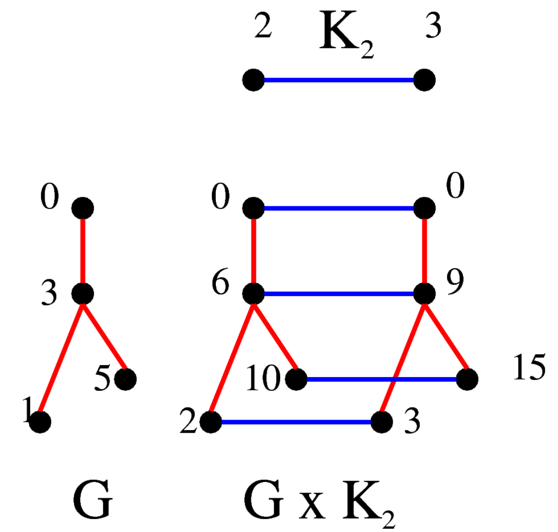


Products of ***sets*** of distributions
S_1 a set of distros on G
S_2 a set of distros on H

$$S_1 \cdot S_2 = \{ D_1 \cdot D_2 : D_1 \in S_1 ; D_2 \in S_2 \}$$

Graham's Conjecture generalized

Graham's Conjecture:

$$\pi(G \times H) \leq \pi(G)\pi(H)$$

Generalization:

$$\pi(G \times H, S_1 \cdot S_2) \leq \pi(G, S_1)\pi(H, S_2)$$

Optimal pebbling

Observation (not obvious):

If we can get from D_1 to D_1' in G and from D_2 to D_2' in H, we can get from $D_1 \cdot D_2$ to $D_1 \cdot D_2'$ in GxH to $D_1' \cdot D_2'$ in GxH

Conclusion (optimal pebbling):

$$\pi^*(G \times H, S_1 \cdot S_2) \leq \pi^*(G, S_1)\pi^*(H, S_2)$$

Graham's Conjecture holds for optimal pebbling in most general setting

Weighted graphs

Edges have weights w

Pebbling moves: remove w pebbles from one vertex, move 1 to adjacent vertex

$\pi(G)$, $\pi(G, D)$ and $\pi(G, S)$ still make sense

GxH also makes sense:

$wt(\{(v,w), (v,w')\}) = wt(\{w, w'\})$

$wt(\{(v,w), (v',w)\}) = wt(\{v, v'\})$

The Good news

Chung: Hypercubes ($K_2 \times K_2 \times \ldots \times K_2$) satisfies Graham's Conjecture for any collection of weights on the edges

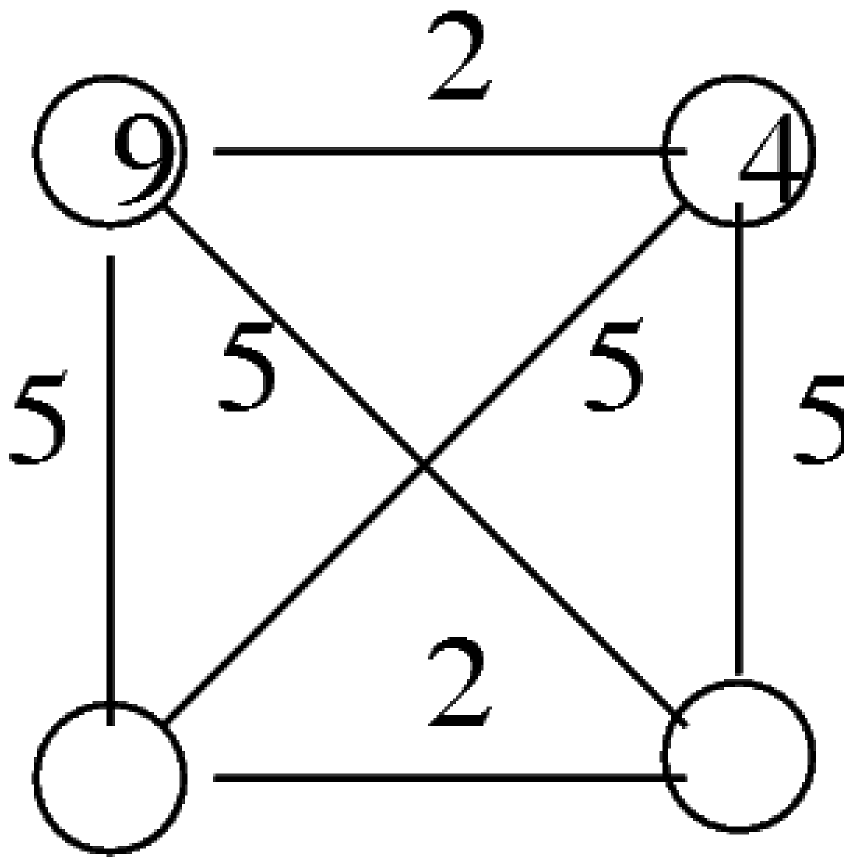
We can focus on complete graphs in most applications

The Bad News

Complete graphs are hard!

The Bad News

- Complete graphs are hard!
- Sjöstrund's Theorem fails
  - 13 pebbles on one vertex can cover K_4, but...



Some specializations

Conjecture 1:

$$\pi(G\times H,(v,w))\leq\pi(G,v)\pi(H,w)$$

Conjecture 2:

$$\pi_{st}(G\times H,(v,w))\leq\pi_s(G,v)\pi_t(H,w)$$

Clearly Conjecture 2 implies Conjecture 1

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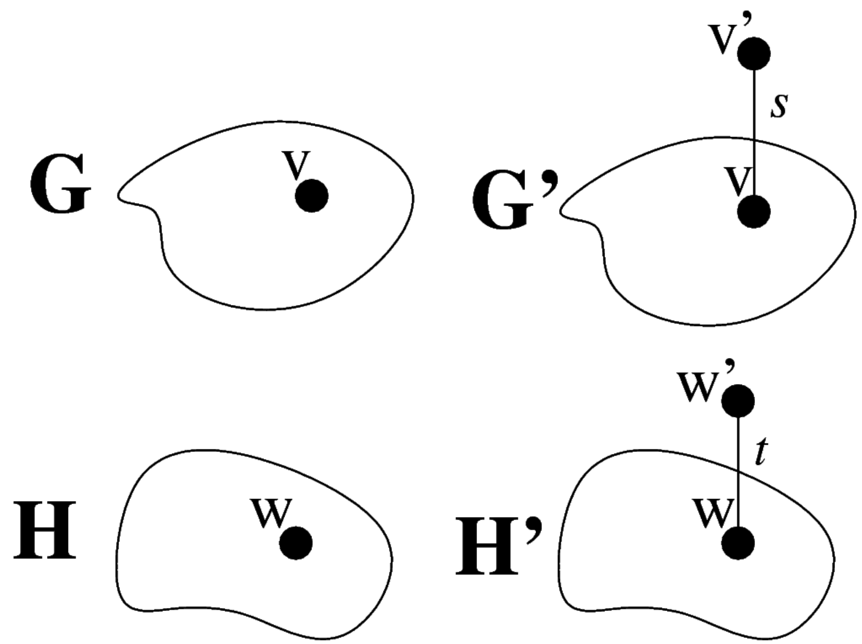
These conjectures are equivalent on weighted graphs

$$\pi(G \times H, (v,w)) \leq \pi(G, v)\, \pi(H, w)$$

implies

$$\pi_{st}(G \times H, (v,w)) \leq \pi_s(G, v)\, \pi_t(H, w)$$



$$\pi(G', v') = \pi_S(G, v)$$

$$\pi(H', w') = \pi_S(H, w)$$

$\pi(G \times H, (v,w)) \leq \pi(G, v)\, \pi(H, w)$ implies $\pi_{st}(G \times H, (v,w)) \leq \pi_s(G, v)\, \pi_t(H, w)$

If st pebbles cannot be moved to (v, w) from D in GxH, then (v', w') cannot be reached from D in G'xH' (delay moves onto {v'}xH' and G'x{w'} as long a possible)

$$\pi_{st}(G \times H, (v, w)) \leq \pi(G' \times H', (v', w'))$$
$$\leq \pi(G', v')\pi(H', w') = \pi_s(G, v)\pi_t(H, w)$$

Implications for regular pebbling

Conjecture 1:

$$\pi(G \times H, (v, w)) \leq \pi(G, v)\pi(H, w)$$

equivalent to Conjecture 1':

$$\pi_{2^{ab}}(G \times H, (v, w)) \leq \pi_{2^a}(G, v)\pi_{2^b}(H, w)$$

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Conjecture 2:

$$\pi_{st}(G \times H, (v, w)) \leq \pi_s(G, v)\pi_t(H, w)$$

equivalent to Conjecture 2':

$$\pi_{st}(G \times H, (v, w)) \leq \pi_s(G, v)\pi_t(H, w)$$

when s and t are odd

Choosing a target

Observation: To reach an unoccupied vertex v in G, we need to put two pebbles on any neighbor of v.

We can choose the target neighbor

If S is a *set* of distributions on G, $\rho(G, S)$ is the number of pebbles needed to reach ***some*** distribution in S

Idea: Develop an induction argument to prove Graham's conjecture

Comparing pebbling numbers

$$\forall D \in S \, \forall D' \in S(G, \pi(G, S))$$

D is reachable from D' by a sequence of pebbling moves

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$\forall D' \in S(G, \rho(G, S)) \, \exists D \in S$

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Properties of ρ(G, S)

$$\rho_1(G) = 1$$

$$\rho_2(G) = |V(G)| + 1$$

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SURPRISE!

Graham's Conjecture fails!

Let H be the trivial graph: $S_H = \{2\delta_v\}$

$$\rho(G \times H, S_1(G) \cdot S_H) = \rho_2(G) = |V(G)| + 1$$

$$\rho(G, S_1(G)) = \rho_1(G) = 1$$

$$\rho(H, S_H) = 2$$