Totally Silver Graphs

Mohammad Ghebleh

Department of Mathematics and Computer Science
Kuwait University

CanaDAM
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This is a joint work with:

Ebad S. Mahmoodian

Department of Mathematical Sciences
Sharif University of Technology
Tehran, Iran
**Silver matrix:**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td>6</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>10</td>
<td>1</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>8</td>
<td>11</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>11</td>
<td>9</td>
<td>7</td>
<td>1</td>
<td>3</td>
<td></td>
</tr>
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<td>11</td>
<td>9</td>
<td>7</td>
<td>10</td>
<td>8</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
### An IMO Problem

**Silver matrix:**

$$
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 1 & 5 & 3 & 6 & 4 \\
8 & 10 & 1 & 6 & 4 & 2 \\
9 & 8 & 11 & 1 & 2 & 5 \\
10 & 11 & 9 & 7 & 1 & 3 \\
11 & 9 & 7 & 10 & 8 & 1 \\
\end{array}
$$

There is no silver matrix of order 1997.

There exist silver matrices of infinitely many different orders.
Silver matrix:

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Silver Matrices

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- There is no silver matrix of order 1997.
- There exist silver matrices of infinitely many different orders.

Theorem (Mahdian, Mahmoodian; 2000)
A silver matrix of order $n$ exists, if and only if $n = 1$ or $2|n$. 
Theorem (G, Goddyn, Mahmoodian, Verdian; 2008)
A silver cube of order $n = 2^a 3^b 5^c$ exists.

Theorem (Ventullo, Khodkar; 2007)
A silver cube of order 7 exists.
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Silver Cubes

Definition

An \((n, d)\)-silver cube is any

\[ c : V(K_n^d) \rightarrow \{0, 1, \ldots, d(n - 1)\} \]

in which every vertex of a diagonal is rainbow.
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Open Problems

- \(K_{11}^3\) (\(K_p^3\) is silver where \(p \geq 11\) is a prime)
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**Open Problems**

- \(K_{11}^3\) (\(K_p^3\) is silver where \(p \geq 11\) is a prime)
- \(Q_{20}\) (\(Q_n\) is not silver where \(4|n\) and \(n \neq 2^t\))
Definition

Given an $r$–regular $G$ and a diagonal $I$,

$$c : V(G) \to \{0, 1, \ldots, r\}$$

is a \textit{silver colouring} if every $v \in I$ is rainbow.
Definition

Given an $r$–regular $G$ and a diagonal $I$,

$$c : V(G) \rightarrow \{0, 1, \ldots, r\}$$

is a silver colouring if every $v \in I$ is rainbow. Totally silver if every $v \in V(G)$ is rainbow.

$G$ is said to be (totally) silver, if it admits a (totally) silver colouring.
Observation

$G$ is totally silver iff $G$ is \textit{domatically full}, i.e. $V(G)$ admits $\delta(G) + 1$ disjoint dominating sets.
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Observation

$G$ is totally silver iff $\chi(G^2) = r + 1$ where $G^2$ is the \textit{square} of $G$. 
Examples of Totally Silver Graphs

```
1  2  3
4  5  1
7  1  6
```

```
6  7  1
3  1  2
1  4  5
```

```
5  1  4
1  6  7
2  3  1
```
Examples of Totally Silver Graphs
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Characterization of Totally Silver Graphs
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$C_{r+1}$

$C_1$

$C_2$

$C_3$
Characterization of Totally Silver Graphs

$C_{r+1}$

$C_1$

$C_2$

$C_3$
Characterization of Totally Silver Graphs

Theorem

Every $r$–regular totally silver graph can be obtained by a sequence of (coloured) 2–switches from a disjoint union of copies of $r + 1$–cliques.
Totally Silver Cubic Graphs

3– and 4–cycles can be reduced.
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5–cycles are forbidden.

So a nontrivial totally silver cubic graph is bridgeless with girth at least 6.

Question
Do there exist totally silver cubic graphs of high girth?
Totally Silver Cubic Graphs

- 3- and 4-cycles can be reduced.
- 5-cycles are forbidden.
- Bridges are forbidden: indeed, every totally silver cubic graph is 3-edge colourable.
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**Question**

Do there exist totally silver cubic graphs of high girth?
An Example with Girth 9

[Diagram of a graph with vertices labeled 1, 2, 3, and edges connecting them in a cycle to form a graph with girth 9.]
An Example with Girth 9
An Example with Girth 9
An Example with Girth 9
An Example with Girth 9
An Example with Girth 9
An Example with Girth 9
An Example with Girth 9
An Example with Girth 9
Thank you!