Schröder numbers, large and small

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Large Schröder numbers

A Schröder path is a path in the plane, starting and ending on the *x*-axis, never going below the *x*-axis, using the steps

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$(2,0)$	flat

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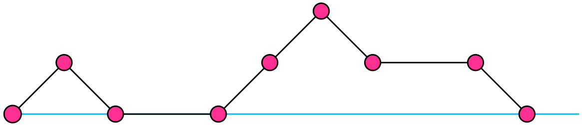
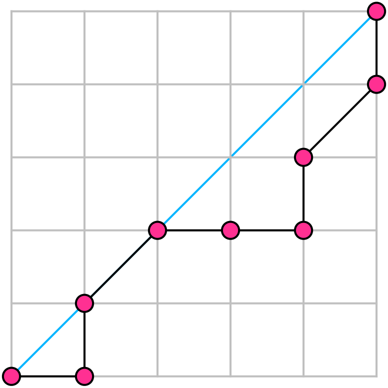


Sometimes it’s convenient to draw a Schröder path in “Cartesian coordinates":



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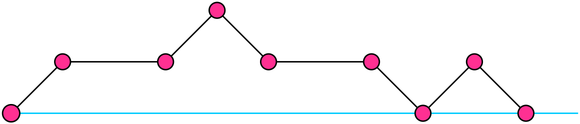

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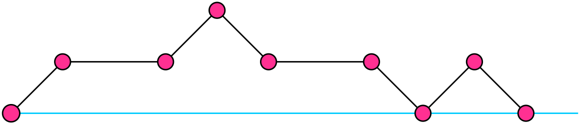

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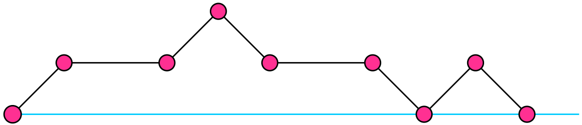

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n	0	1	2	3	4	5	6	7	8	9
r_n	1	2	6	22	90	394	1806	8558	41586	206098
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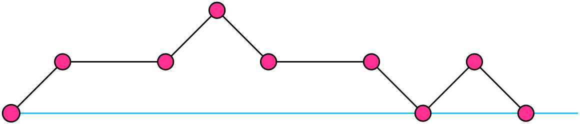

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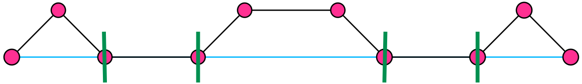
Theorem. For $n > 0$, $r_n = 2s_n$.

Generating function proof #1

Let $R(x) = \sum_{n=0}^{\infty} r_n x^n$ and let $S(x) = \sum_{n=0} s_n x^n$.

Every Schröder path can be uniquely decomposed into *prime* Schröder paths:



Each prime is either a flat step or an up step followed by a Schröder path followed by a down step, so the generating function for prime Schröder paths is $x + xR(x)$. Therefore,

$$R(x) = \sum_{k=0}^{\infty} (x + xR(x))^k = \frac{1}{1 - x - xR(x)}.$$

and similarly

$$S(x) = \frac{1}{1 - xR(x)}.$$

The first equation is a quadratic, which may be written

$$xR(x)^2 + (x-1)R(x) + 1 = 0.$$

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so $R(x) = 2S(x) - 1$.

Generating function proof #2

We rewrite

$$R(x) = \sum_{k=0}^{\infty} (x + xR(x))^k = \frac{1}{1 - x - xR(x)}$$

as

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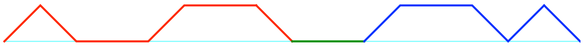
and thus

$$\begin{aligned} R(x) &= \frac{1 + xR(x)}{1 - xR(x)} \\ &= \frac{1}{1 - xR(x)} + \frac{xR(x)}{1 - xR(x)} \\ &= \frac{1}{1 - xR(x)} + \left(\frac{1}{1 - xR(x)} - 1\right) = 2S(x) - 1. \end{aligned}$$

Bijective proof

We find a bijection from Schröder paths with at least one flat step on the x-axis to small Schröder paths (Schröder paths with no flat steps on the x-axis).

We can factor a Schröder path with at least one flat step on the x-axis as PFQ, where F is the last flat step, so Q has no flat steps on the x-axis:



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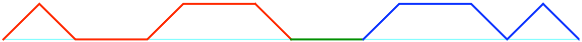
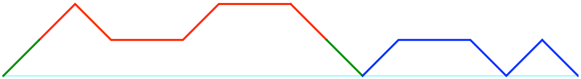


We replace the path with $UPDQ$ where U is an up step and D is a down step:



Schröder polynomials

Instead of just counting Schröder paths, we can weight them by $\alpha^{\#\text{flat steps}}$. We get Schröder polynomials $r_n(\alpha)$ and $s_n(\alpha)$, with $r_n(1) = r_n$ and $s_n(1) = s_n$. Everything that we've done so far extends to $r_n(\alpha)$ and $s_n(\alpha)$. With $R(x) = \sum_{n=0}^{\infty} r_n(\alpha)x^n$ and $S(x) = \sum_{n=0}^{\infty} s_n(\alpha)x^n$, we have

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so

$$R(x) = \frac{1 - \alpha x - \sqrt{(1-\alpha x)^2 - 4x}}{2x}$$
$$S(x) = \frac{1 + \alpha x - \sqrt{(1-\alpha x)^2 - 4x}}{2x(1+\alpha)}$$

and $r_n(\alpha) = (1+\alpha)s_n(\alpha)$ for $n > 0$.

We also have explicit formulas

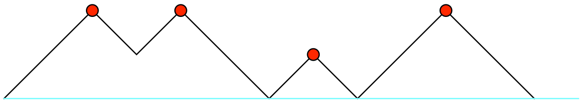
$$r_n(\alpha) = \sum_{k=0}^{n} \frac{1}{n-k+1} \binom{2n-2k}{n-k} \binom{2n-k}{k} \alpha^k$$

$$= \sum_{k=0}^{n} C_{n-k} \binom{2n-k}{k} \alpha^k$$

$$s_n(\alpha) = \sum_{k=0}^{n-1} \frac{1}{n+1} \binom{n-1}{k} \binom{2n-k}{n} \alpha^k$$

Narayana numbers

The Narayana number $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ is the number of Dyck paths of semilength n with k peaks.



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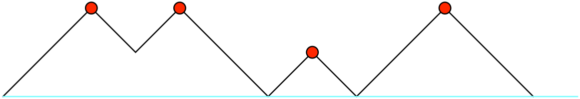
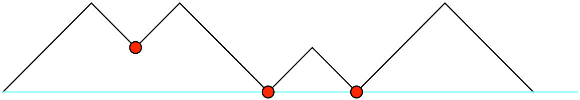


A Dyck path with k peaks has $k-1$ valleys, so $N(n,k)$ is also the number of Dyck paths with $k-1$ valleys.

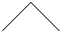


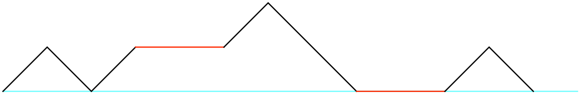
Let

$$\overline{N}_n(\alpha) = \sum_{k=1}^{n} N(n,k)\alpha^{k-1} \quad \text{and} \quad N_n(\alpha) = \sum_{k=1}^{n} N(n,k)\alpha^{k},$$

so that $N_n(\alpha) = \alpha\overline{N}_n(\alpha)$.

Proof #4

To any Schröder path we associate a Dyck path by replacing each flat step with a peak: 



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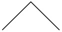
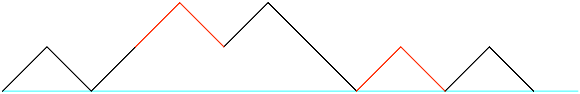
To any Schröder path we associate a Dyck path by replacing each flat step with a peak: 



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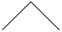
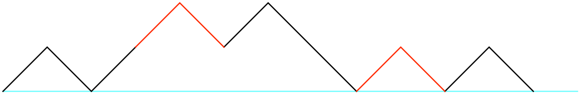
To any Schröder path we associate a Dyck path by replacing each flat step with a peak: 



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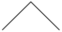
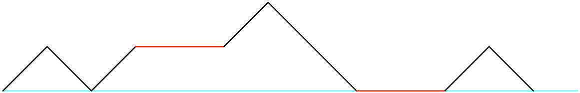
To any Schröder path we associate a Dyck path by replacing each flat step with a peak: 



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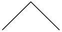
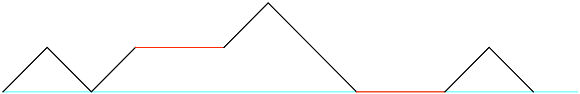
To any Schröder path we associate a Dyck path by replacing each flat step with a peak: 



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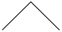
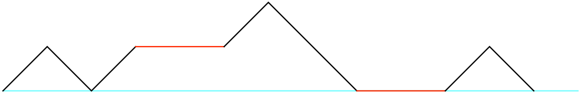
To any Schröder path we associate a Dyck path by replacing each flat step with a peak: 



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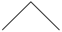
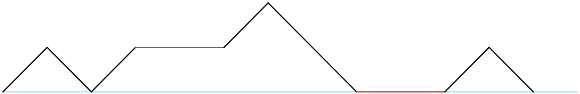
To any Schröder path we associate a Dyck path by replacing each flat step with a peak: 



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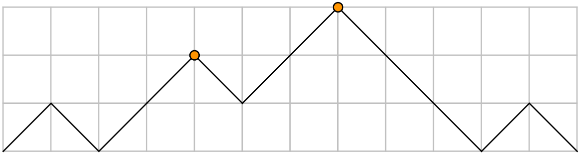
High peaks

A high peak is a peak that is at height greater than 1:



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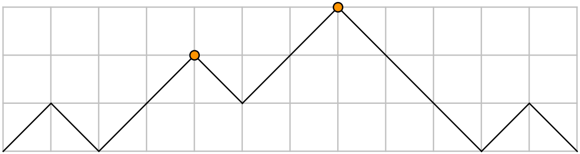

Let $\widetilde{N}_n(\alpha)$ count Dyck paths of semilength n by high peaks.

We can get small Schröder paths from Dyck paths by replacing some of the high peaks with flat steps, so as before, we get $s_n(\alpha) = \widetilde{N}_n(1 + \alpha)$.

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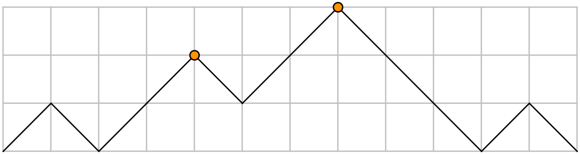

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Since we already know that $s_n(\alpha) = \overline{N}_n(1 + \alpha)$, we have $\widetilde{N}_n(\alpha) = \overline{N}_n(\alpha)$.

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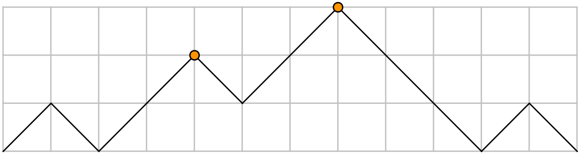

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Since we already know that $s_n(\alpha) = \overline{N}_n(1+\alpha)$, we have $\widetilde{N}_n(\alpha) = \overline{N}_n(\alpha)$. Is there a bijective proof?

A bijective proof was given by Emeric Deutsch, *A bijection on Dyck paths and its consequences*, Discrete Math. 179 (1998), 253–256.

Deutsch also stated, "Sulanke [private communication] has constructed another bijection on Dyck paths from which one obtains the equidistribution of the parameters (i) the number of high peaks and (ii) the number of valleys. Namely, for each path raise the horizontal axis two units and let the high peaks become the valleys of the image path."

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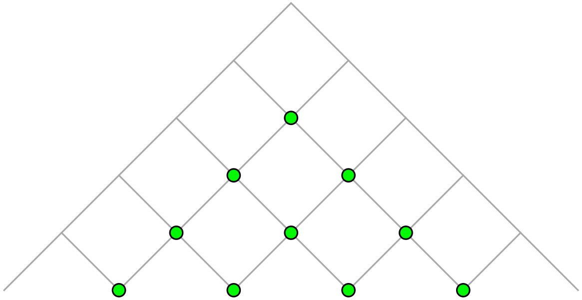
What does this mean?

Key observation: A Dyck path is determined by the positions of its valleys, and also by the positions of its high peaks:



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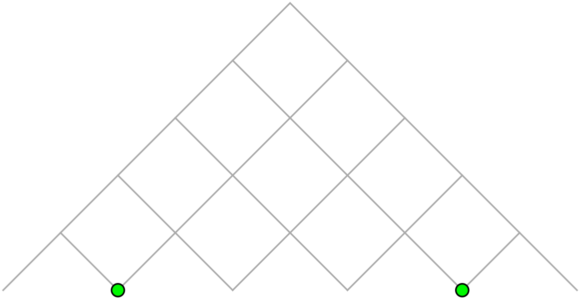

A choice of positions for valleys

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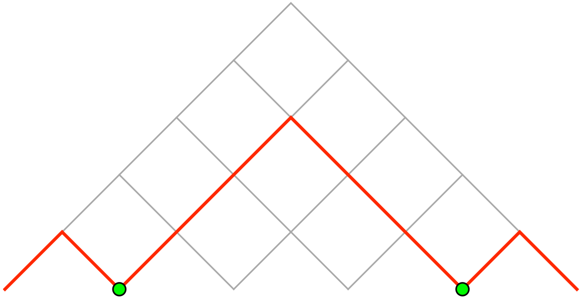

The path with chosen valleys

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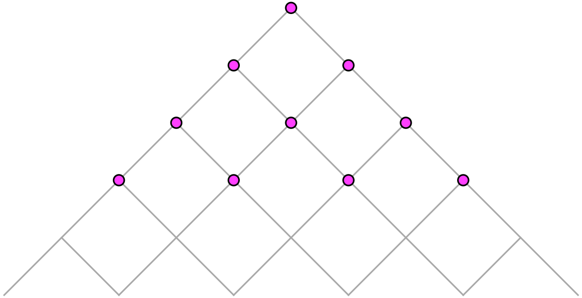

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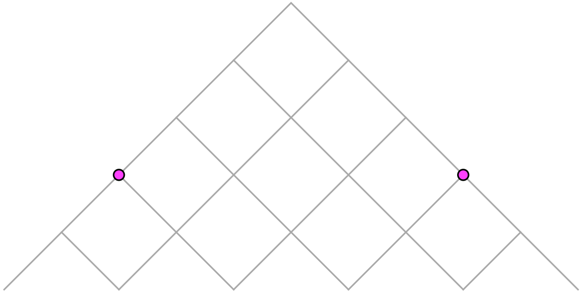

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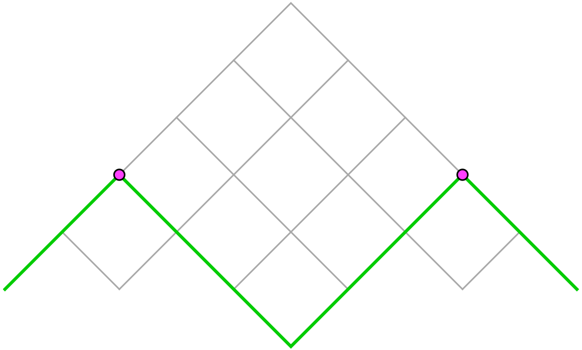

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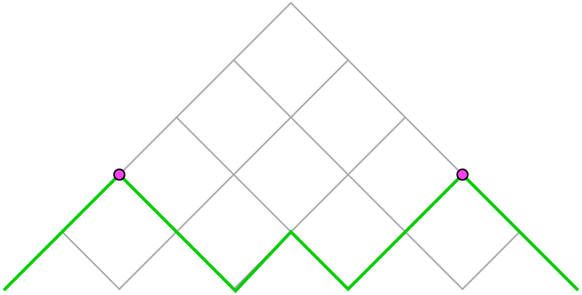
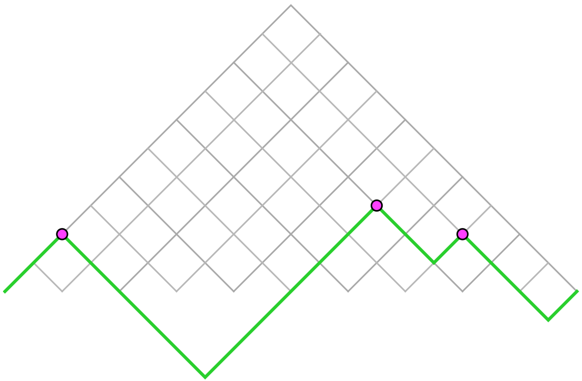


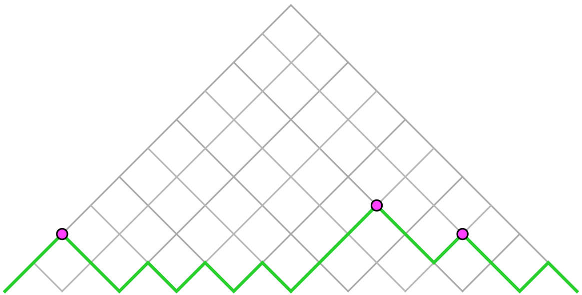
A Dyck path with the chosen high peaks

The last step with a bigger example:



A path with the chosen peaks

The last step with a bigger example:



A Dyck path with the chosen high peaks

Motzkin and Riordan paths

A Motzkin path is a path in the plane, starting and ending on the x-axis, never going below the x-axis, using the steps

$(1, 1)$	up
$(1, -1)$	down
$(1, 0)$	flat

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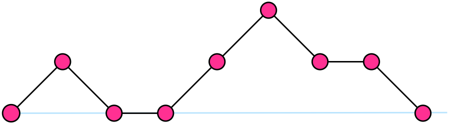

A Riordan path is a Motzkin path with no flat steps on the x-axis.

Let M_n be the number of Motzkin paths of length n and let J_n be the number of Riordan paths of length n.

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$$\sum_{n=0}^{\infty} M_n x^n = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}$$

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n	0	1	2	3	4	5	6	7	8	9	10
M_n	1	1	2	4	9	21	51	127	323	835	2188
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Theorem. $M_n = J_n + J_{n+1}$.

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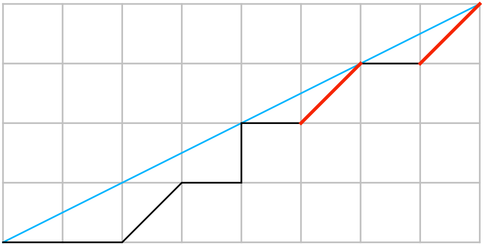
Theorem. $M_n = J_n + J_{n+1}$.
Proof: The same as before.

Generalized Schröder paths

It is convenient to use “Cartesian coordinates". We look at paths using north, east, and northeast steps that stay below the line $x = my$ for some integer m:



The role of flat steps on the x-axis is now played by diagonal steps that end on the line $x = my$.

Theorem. Let r_n be the number of paths from $(0, 0)$ to (mn, n) and let s_n be the number of these paths with no diagonal steps ending on the line $x = my$. Then for $n > 0$, $r_n = 2s_n$

Theorem. Let r_n be the number of paths from $(0,0)$ to (mn, n) and let s_n be the number of these paths with no diagonal steps ending on the line $x = my$. Then for $n > 0$, $r_n = 2s_n$

Bijective proof. The same as in the case $m = 1$.

Generating function proof (sketch). Let $R(x) = \sum_{n=0}^{\infty} r_n x^n$ and $S(x) = \sum_{n=0}^{\infty} s_n x^n$. Then

$$R(x) = \frac{1}{1 - xR(x)^{m-1} - xR(x)^m}$$

and

$$S(x) = \frac{1}{1 - xR(x)^m}.$$

So

$$R(x) = \frac{1 + xR(x)^m}{1 - xR(x)^m} = 2S(x) - 1.$$