

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

# ON MINIMUM AND MAXIMUM VALUES OF $\gamma$ -LABELINGS OF GRAPHS

Grady Bullington   Linda Eroh   Steven J. Winters  
University of Wisconsin Oshkosh

CANAdian Discrete and Algorithmic Mathematics  
Conference, May 25-28, 2009

# OUTLINE

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

1 DEFINITIONS

2 EXAMPLES

3 COMPLETE BIPARTITE

4 SPECTRA OF  $K_{n,n}$

5 PRODUCTS OF CYCLES

# OUTLINE

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

1 DEFINITIONS

2 EXAMPLES

3 COMPLETE BIPARTITE

4 SPECTRA OF  $K_{n,n}$

5 PRODUCTS OF CYCLES

# OUTLINE

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

1 DEFINITIONS

2 EXAMPLES

3 COMPLETE BIPARTITE

4 SPECTRA OF  $K_{n,n}$

5 PRODUCTS OF CYCLES

# OUTLINE

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

1 DEFINITIONS

2 EXAMPLES

3 COMPLETE BIPARTITE

4 SPECTRA OF  $K_{n,n}$

5 PRODUCTS OF CYCLES

# OUTLINE

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

1 DEFINITIONS

2 EXAMPLES

3 COMPLETE BIPARTITE

4 SPECTRA OF  $K_{n,n}$

5 PRODUCTS OF CYCLES

# $\gamma$ -LABELINGS

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

A  $\gamma$ -labeling of a graph  $G$  with  $m$  edges is a one-to-one function from  $V(G)$  to  $\{0, 1, \dots, m\}$ .

A  $\gamma$ -labeling exists for a graph if the order is at most one more than the size. Any connected graph has a  $\gamma$ -labeling.

A  $\gamma$ -labeling induces a labeling of the edges. The edge  $uv$  is labeled with  $|label(u) - label(v)|$ .

G. Chartrand, D. Erwin, D. VanderJagt, P. Zhang.  
 *$\gamma$ -labelings of graphs*. Bulletin of the ICA 44(2005), 51 - 68.

# $\gamma$ -LABELINGS

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

A  $\gamma$ -labeling of a graph  $G$  with  $m$  edges is a one-to-one function from  $V(G)$  to  $\{0, 1, \dots, m\}$ .

A  $\gamma$ -labeling exists for a graph if the order is at most one more than the size. Any connected graph has a  $\gamma$ -labeling.

A  $\gamma$ -labeling induces a labeling of the edges. The edge  $uv$  is labeled with  $|label(u) - label(v)|$ .

G. Chartrand, D. Erwin, D. VanderJagt, P. Zhang.  
 *$\gamma$ -labelings of graphs*. Bulletin of the ICA 44(2005), 51 - 68.

# $\gamma$ -LABELINGS

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

A  $\gamma$ -labeling of a graph  $G$  with  $m$  edges is a one-to-one function from  $V(G)$  to  $\{0, 1, \dots, m\}$ .

A  $\gamma$ -labeling exists for a graph if the order is at most one more than the size. Any connected graph has a  $\gamma$ -labeling.

A  $\gamma$ -labeling induces a labeling of the edges. The edge  $uv$  is labeled with  $|label(u) - label(v)|$ .

G. Chartrand, D. Erwin, D. VanderJagt, P. Zhang.  
 *$\gamma$ -labelings of graphs*. Bulletin of the ICA 44(2005), 51 - 68.

# $\gamma$ -LABELINGS

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

A  $\gamma$ -labeling of a graph  $G$  with  $m$  edges is a one-to-one function from  $V(G)$  to  $\{0, 1, \dots, m\}$ .

A  $\gamma$ -labeling exists for a graph if the order is at most one more than the size. Any connected graph has a  $\gamma$ -labeling.

A  $\gamma$ -labeling induces a labeling of the edges. The edge  $uv$  is labeled with  $|label(u) - label(v)|$ .

G. Chartrand, D. Erwin, D. VanderJagt, P. Zhang.  
 *$\gamma$ -labelings of graphs*. Bulletin of the ICA **44**(2005), 51 - 68.

# MAXIMUM AND MINIMUM LABELINGS

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

The **value** of a  $\gamma$ -labeling is the sum of the induced labels of the edges.

$val_{max}(G)$  = maximum value over all  $\gamma$ -labelings of  $G$   
 $val_{min}(G)$  = minimum value over all  $\gamma$ -labelings of  $G$

The **spectrum** of  $G$  is the set of all possible values of  $\gamma$ -labelings of  $G$ .

# MAXIMUM AND MINIMUM LABELINGS

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

The **value** of a  $\gamma$ -labeling is the sum of the induced labels of the edges.

$val_{max}(G)$  = maximum value over all  $\gamma$ -labelings of  $G$   
 $val_{min}(G)$  = minimum value over all  $\gamma$ -labelings of  $G$

The **spectrum** of  $G$  is the set of all possible values of  $\gamma$ -labelings of  $G$ .

# MAXIMUM AND MINIMUM LABELINGS

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

The **value** of a  $\gamma$ -labeling is the sum of the induced labels of the edges.

$val_{max}(G)$  = maximum value over all  $\gamma$ -labelings of  $G$   
 $val_{min}(G)$  = minimum value over all  $\gamma$ -labelings of  $G$

The **spectrum** of  $G$  is the set of all possible values of  $\gamma$ -labelings of  $G$ .

# BANDWIDTH

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

I recently learned that this is very similar to the **bandwidth** of a labeling.

If the vertices of  $G$  are labeled with  $\{1, 2, \dots, |V(G)|\}$ , the **bandwidth** is the sum of the induced edge labels.

# EXAMPLE $Val_{min}(G)$

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.

BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

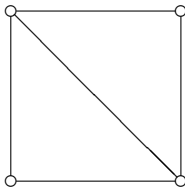
EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

A  $\gamma$ -labeling will use labels from  $\{0, 1, 2, 3, 4, 5\}$ .



# EXAMPLE $Val_{min}(G)$

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.

BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

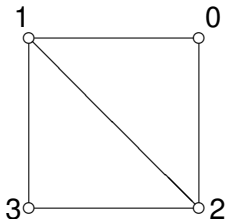
EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

A  $\gamma$ -labeling will use labels from  $\{0, 1, 2, 3, 4, 5\}$ .



# EXAMPLE $Val_{min}(G)$

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.

BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

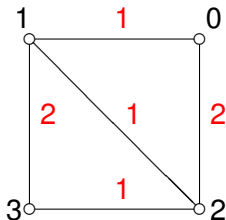
EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

A  $\gamma$ -labeling will use labels from  $\{0, 1, 2, 3, 4, 5\}$ .



# EXAMPLE $Val_{min}(G)$

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.

BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

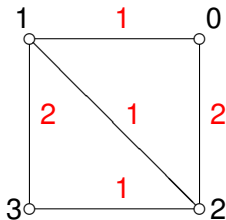
EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

A  $\gamma$ -labeling will use labels from  $\{0, 1, 2, 3, 4, 5\}$ .  
 $val_{min}(G) = 7$ .



# EXAMPLE $Val_{max}(G)$

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.

BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

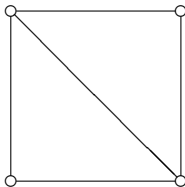
EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

A  $\gamma$ -labeling will use labels from  $\{0, 1, 2, 3, 4, 5\}$ .



# EXAMPLE $Val_{max}(G)$

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.

BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

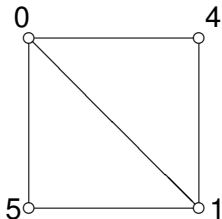
EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

A  $\gamma$ -labeling will use labels from  $\{0, 1, 2, 3, 4, 5\}$ .



# EXAMPLE $Val_{max}(G)$

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

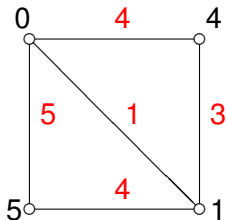
EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

A  $\gamma$ -labeling will use labels from  $\{0, 1, 2, 3, 4, 5\}$ .



# EXAMPLE $Val_{max}(G)$

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.

BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

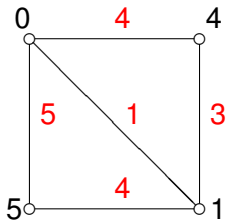
EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

A  $\gamma$ -labeling will use labels from  $\{0, 1, 2, 3, 4, 5\}$ .  
 $val_{max}(G) = 17$ .



# $val_{min}(K_{n_1, n_2})$

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

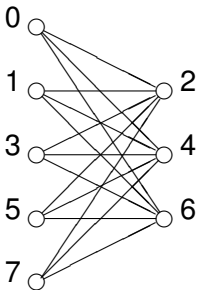
COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

For  $n_1 \geq n_2$ ,

$$val_{min}(K_{n_1, n_2}) = \frac{n_2(2n_2^2+1)}{3} + (n_1 - n_2)n_2^2 + \left\lfloor \frac{(n_1 - n_2)^2}{4} \right\rfloor n_2$$



# $val_{min}(K_{n_1, n_2})$

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

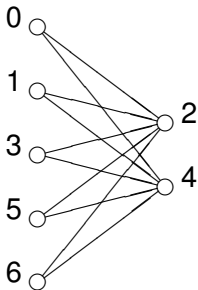
COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

For  $n_1 \geq n_2$ ,

$$val_{min}(K_{n_1, n_2}) = \frac{n_2(2n_2^2+1)}{3} + (n_1 - n_2)n_2^2 + \left\lfloor \frac{(n_1 - n_2)^2}{4} \right\rfloor n_2$$



$$val_{min}(K_{n_1, n_2, \dots, n_r})$$

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

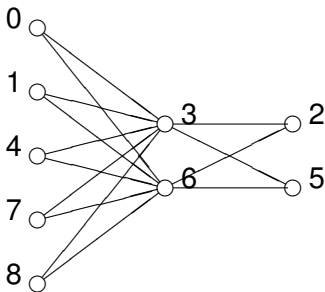
EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

A similar labeling produces the minimum value for a complete multipartite graph. (Some edges left out for clarity)



$$val_{min}(K_{n_1, n_2, \dots, n_r})$$

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

A similar labeling produces the minimum value for a complete multipartite graph.

Sketch of proof: WLOG, labels are consecutive and start at 0. Given any labeling with these labels, it can be converted into this labeling by a series of swaps which does not increase the value.

$$val_{min}(K_{n_1, n_2, \dots, n_r})$$

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

A similar labeling produces the minimum value for a complete multipartite graph.

Sketch of proof: WLOG, labels are consecutive and start at 0. Given any labeling with these labels, it can be converted into this labeling by a series of swaps which does not increase the value.

# $val_{max}(K_{n_1, n_2})$

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

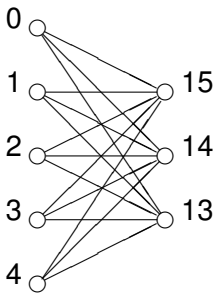
EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n, n}$

PRODUCTS OF  
CYCLES

$$val_{max}(K_{n_1, n_2}) = n_1 n_2 (n_1 n_2 - \frac{1}{2} n_1 - \frac{1}{2} n_2 + 1)$$



# PARTIAL RESULTS ON SPECTRA

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

*For every even positive integer  $n$ , every number in the spectrum of  $K_{n,n}$  is even.*

Sketch of proof: First, we show that any  $\gamma$ -labeling of  $K_{n,n}$  can be converted into any other  $\gamma$ -labeling by a series of moves, where each move is one of the following:

add or subtract 1 from a single label

swap two consecutive labels used in opposite partite sets

Then we show that each of these moves changes the value by an even number.

# PARTIAL RESULTS ON SPECTRA

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

*For every even positive integer  $n$ , every number in the spectrum of  $K_{n,n}$  is even.*

Sketch of proof: First, we show that any  $\gamma$ -labeling of  $K_{n,n}$  can be converted into any other  $\gamma$ -labeling by a series of moves, where each move is one of the following:

- add or subtract 1 from a single label

- swap two consecutive labels used in opposite partite sets

Then we show that each of these moves changes the value by an even number.

# PARTIAL RESULTS ON SPECTRA

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

*For every even positive integer  $n$ , every even integer from  $\frac{n(2n^2+1)}{3}$  to  $\frac{n^2(2n^2-n+2)}{4}$  is in the spectrum of  $K_{n,n}$ .*

From previous results,  $val_{min}(K_{n,n}) = \frac{n(2n^2+1)}{3}$  and  $val_{max}(K_{n,n}) = n^2(n^2 - n + 1)$ .

# PARTIAL RESULTS ON SPECTRA

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

*For any odd integer  $n \geq 3$ , every integer from  $\frac{n(n+1)(3n-1)}{4}$  to  $\frac{n(2n^3+n^2-2n+3)}{4}$  is in the spectrum of  $K_{n,n}$ .*

From previous results,  $val_{min}(K_{n,n}) = \frac{n(2n^2+1)}{3}$  and  $val_{max}(K_{n,n}) = n^2(n^2 - n + 1)$ .

# PARTIAL RESULTS ON SPECTRA

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

*For any integer  $n \geq 2$ , the value  $\frac{n(2n^2+1)}{3} + 1$  is not in the spectrum of  $K_{n,n}$ . Furthermore, the values  $n^2(n^2 - n + 1) - i$ , for  $1 \leq i \leq n - 1$ , are not in the spectrum of  $K_{n,n}$ .*

# $Val_{max}(C_a \times C_b)$

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

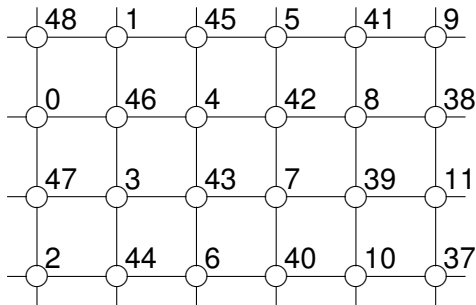
EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

If  $a$  and  $b$  are even integers,  $a \geq 4$  and  $b \geq 4$ , then  
 $val_{max}(C_a \times C_b) = ab(3ab + 2)$ .



# $Val_{max}(C_a \times C_b)$

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

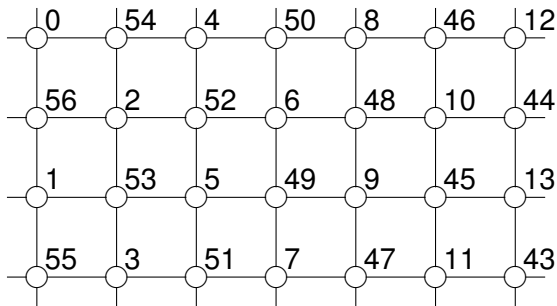
COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

If  $a \geq 3$  is odd and  $b \geq 4$  is even, then

$$val_{max}(C_a \times C_b) = 3a^2b^2 - ab^2 + 2ab - \frac{1}{2}b^2 - b.$$



# $Val_{max}(C_a \times C_b)$

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

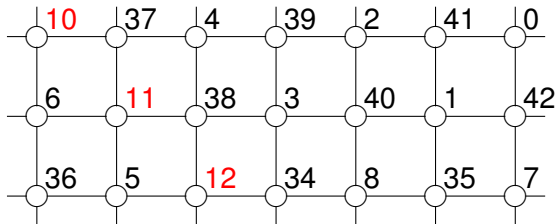
COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

If  $a \geq 3$  and  $b \geq 3$  are odd integers, then

$$val_{max}(C_a \times C_b) = 3a^2b^2 - ab^2 - a^2b + 2ab - a - b - \frac{1}{2}(a^2 + b^2).$$



# $Val_{min}(C_a \times C_b)$

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

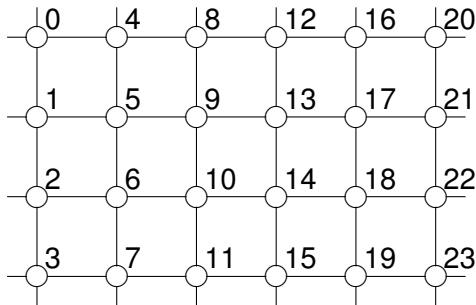
EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

If  $a \geq b \geq 3$  are integers, then  
 $val_{min}(C_a \times C_b) = 2a(b - 1) + 2b^2(a - 1)$ .



# A SAMPLING OF OTHER RESULTS ON $\gamma$ -LABELING

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

G. Chartrand, D. Erwin, D.W. VanderJagt, and P. Zhang found the spectrum of stars  $K_{1,n}$  and the maximum and minimum value of paths, cycles, and complete graphs.

C.M. de Fonseca, Varaporn Saenpholphat and Ping Zhang determined the spectra of paths, cycles, and complete graphs.

# A SAMPLING OF OTHER RESULTS ON $\gamma$ -LABELING

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

G. Chartrand, D. Erwin, D.W. VanderJagt, and P. Zhang found the spectrum of stars  $K_{1,n}$  and the maximum and minimum value of paths, cycles, and complete graphs.

C.M. de Fonseca, Varaporn Saenpholphat and Ping Zhang determined the spectra of paths, cycles, and complete graphs.

# A SAMPLING OF OTHER RESULTS ON $\gamma$ -LABELING

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

Futaba Okamoto, Ping Zhang, and Varaporn Saenpholpat defined balanced  $\gamma$ -labeling of digraphs and characterized which digraphs have balanced  $\gamma$ -labelings.

Grady Bullington showed that the Connell sum sequence is a sharp upper bound on the maximum value of a  $\gamma$ -labeling of a graph of given order.

S.M. Hedge and P. Shankaran defined and studied analogous results for edge sum labeling.

# A SAMPLING OF OTHER RESULTS ON $\gamma$ -LABELING

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

Futaba Okamoto, Ping Zhang, and Varaporn Saenpholpat defined balanced  $\gamma$ -labeling of digraphs and characterized which digraphs have balanced  $\gamma$ -labelings.

Grady Bullington showed that the Connell sum sequence is a sharp upper bound on the maximum value of a  $\gamma$ -labeling of a graph of given order.

S.M. Hedge and P. Shankaran defined and studied analogous results for edge sum labeling.

# A SAMPLING OF OTHER RESULTS ON $\gamma$ -LABELING

ON MINIMUM  
AND  
MAXIMUM  
VALUES OF  
 $\gamma$ -LABELINGS  
OF GRAPHS

G.  
BULLINGTON,  
L. EROH\*, S.  
J. WINTERS

DEFINITIONS

EXAMPLES

COMPLETE  
BIPARTITE

SPECTRA OF  
 $K_{n,n}$

PRODUCTS OF  
CYCLES

Futaba Okamoto, Ping Zhang, and Varaporn Saenpholpat defined balanced  $\gamma$ -labeling of digraphs and characterized which digraphs have balanced  $\gamma$ -labelings.

Grady Bullington showed that the Connell sum sequence is a sharp upper bound on the maximum value of a  $\gamma$ -labeling of a graph of given order.

S.M. Hedge and P. Shankaran defined and studied analogous results for edge sum labeling.