

Crossings, Colorings, and Cliques

Daniel W. Cranston

DIMACS, Rutgers and Bell Labs
and Virginia Commonwealth University
dcransto@dimacs.rutgers.edu

Joint with Mike Albertson and Jacob Fox.

On Graphs with Crossings, CanaDAM
28 May 2009

Crossing number and Albertson's Conjecture

Def. Crossing number of a graph G , $cr(G)$: minimum number of crossings in a (plane) drawing of G .

Crossing number and Albertson's Conjecture

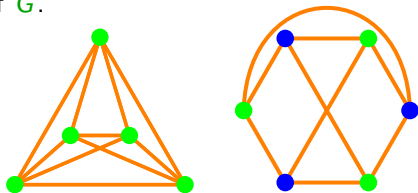
Def. **Crossing number** of a graph G , $\text{cr}(G)$: minimum number of crossings in a (plane) drawing of G .

e.g., $\text{cr}(K_5) = 1$, $\text{cr}(K_{3,3}) = 1$,
 $\text{cr}(G) = 0$ for all planar G .

Crossing number and Albertson's Conjecture

Def. **Crossing number** of a graph G , $cr(G)$: minimum number of crossings in a (plane) drawing of G .

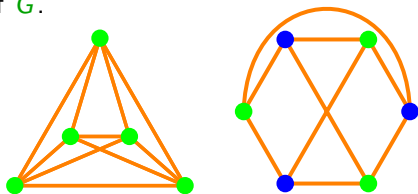
e.g., $cr(K_5) = 1$, $cr(K_{3,3}) = 1$,
 $cr(G) = 0$ for all planar G .



Crossing number and Albertson's Conjecture

Def. **Crossing number** of a graph G , $\text{cr}(G)$: minimum number of crossings in a (plane) drawing of G .

e.g., $\text{cr}(K_5) = 1$, $\text{cr}(K_{3,3}) = 1$,
 $\text{cr}(G) = 0$ for all planar G .

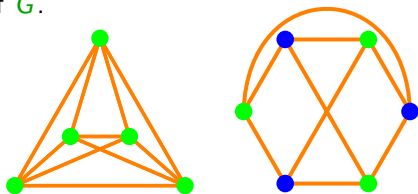


Conj. [Albertson '07] If $\chi(G) = r$, then $\text{cr}(G) \geq \text{cr}(K_r)$.

Crossing number and Albertson's Conjecture

Def. **Crossing number** of a graph G , $\text{cr}(G)$: minimum number of crossings in a (plane) drawing of G .

e.g., $\text{cr}(K_5) = 1$, $\text{cr}(K_{3,3}) = 1$,
 $\text{cr}(G) = 0$ for all planar G .



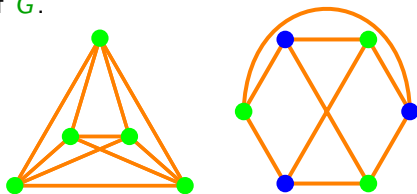
Conj. [Albertson '07] If $\chi(G) = r$, then $\text{cr}(G) \geq \text{cr}(K_r)$.

r	$\text{cr}(K_r)$	Albertson's Conjecture
≤ 4	0	trivial

Crossing number and Albertson's Conjecture

Def. **Crossing number** of a graph G , $cr(G)$: minimum number of crossings in a (plane) drawing of G .

e.g., $cr(K_5) = 1$, $cr(K_{3,3}) = 1$,
 $cr(G) = 0$ for all planar G .



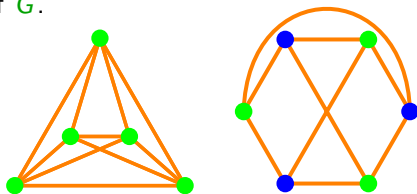
Conj. [Albertson '07] If $\chi(G) = r$, then $cr(G) \geq cr(K_r)$.

r	$cr(K_r)$	Albertson's Conjecture
≤ 4	0	trivial
5	1	4 Color Theorem

Crossing number and Albertson's Conjecture

Def. **Crossing number** of a graph G , $cr(G)$: minimum number of crossings in a (plane) drawing of G .

e.g., $cr(K_5) = 1$, $cr(K_{3,3}) = 1$,
 $cr(G) = 0$ for all planar G .



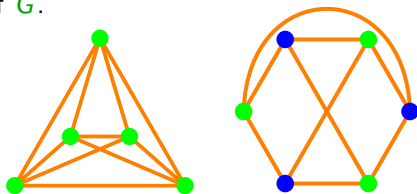
Conj. [Albertson '07] If $\chi(G) = r$, then $cr(G) \geq cr(K_r)$.

r	$cr(K_r)$	Albertson's Conjecture
≤ 4	0	trivial
5	1	4 Color Theorem
6	3	not hard, done

Crossing number and Albertson's Conjecture

Def. **Crossing number** of a graph G , $cr(G)$: minimum number of crossings in a (plane) drawing of G .

e.g., $cr(K_5) = 1$, $cr(K_{3,3}) = 1$,
 $cr(G) = 0$ for all planar G .



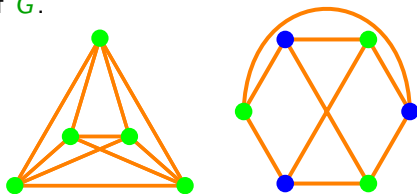
Conj. [Albertson '07] If $\chi(G) = r$, then $cr(G) \geq cr(K_r)$.

r	$cr(K_r)$	Albertson's Conjecture
≤ 4	0	trivial
5	1	4 Color Theorem
6	3	not hard, done
7	9	open

Crossing number and Albertson's Conjecture

Def. **Crossing number** of a graph G , $cr(G)$: minimum number of crossings in a (plane) drawing of G .

e.g., $cr(K_5) = 1$, $cr(K_{3,3}) = 1$,
 $cr(G) = 0$ for all planar G .



Conj. [Albertson '07] If $\chi(G) = r$, then $cr(G) \geq cr(K_r)$.

r	$cr(K_r)$	Albertson's Conjecture
≤ 4	0	trivial
5	1	4 Color Theorem
6	3	not hard, done
7	9	open

Prop. If $\chi(G) = 7$, then $cr(G) \geq 7$.

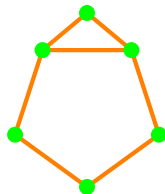
Critical Graphs and Proof of the Prop.

Def. G is a **critical graph** iff $\forall e \in E(G): \chi(G - e) < \chi(G)$.

Critical Graphs and Proof of the Prop.

Def. G is a **critical graph** iff $\forall e \in E(G): \chi(G - e) < \chi(G)$.

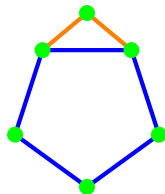
Obs. Every G contains a critical subgraph H with $\chi(H) = \chi(G)$.



Critical Graphs and Proof of the Prop.

Def. G is a **critical graph** iff $\forall e \in E(G): \chi(G - e) < \chi(G)$.

Obs. Every G contains a critical subgraph H with $\chi(H) = \chi(G)$.

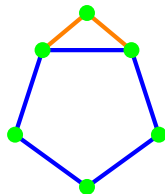


Critical Graphs and Proof of the Prop.

Def. G is a **critical graph** iff $\forall e \in E(G): \chi(G - e) < \chi(G)$.

Obs. Every G contains a critical subgraph H with $\chi(H) = \chi(G)$.

Obs. If G is r -critical, then $\delta(G) \geq r - 1$.



Critical Graphs and Proof of the Prop.

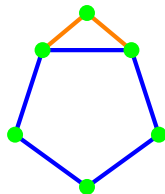
Def. G is a **critical graph** iff $\forall e \in E(G): \chi(G - e) < \chi(G)$.

Obs. Every G contains a critical subgraph H with $\chi(H) = \chi(G)$.

Obs. If G is r -critical, then $\delta(G) \geq r - 1$.

Prop. If $\chi(G) = 7$, then $\text{cr}(G) \geq 7$.

Pf. Assume G is 7-critical and $K_7 \not\subseteq G$.



Critical Graphs and Proof of the Prop.

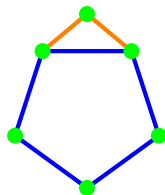
Def. G is a **critical graph** iff $\forall e \in E(G): \chi(G - e) < \chi(G)$.

Obs. Every G contains a critical subgraph H with $\chi(H) = \chi(G)$.

Obs. If G is r -critical, then $\delta(G) \geq r - 1$.

Prop. If $\chi(G) = 7$, then $\text{cr}(G) \geq 7$.

Pf. Assume G is 7-critical and $K_7 \not\subseteq G$.
Since $\chi(G) = 7$, $\delta(G) \geq 6$; so $m \geq \frac{6n}{2} = 3n$.



Critical Graphs and Proof of the Prop.

Def. G is a **critical graph** iff $\forall e \in E(G): \chi(G - e) < \chi(G)$.

Obs. Every G contains a critical subgraph H with $\chi(H) = \chi(G)$.

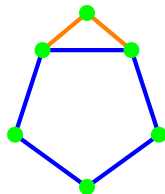
Obs. If G is r -critical, then $\delta(G) \geq r - 1$.

Prop. If $\chi(G) = 7$, then $\text{cr}(G) \geq 7$.

Pf. Assume G is 7-critical and $K_7 \not\subseteq G$.

Since $\chi(G) = 7$, $\delta(G) \geq 6$; so $m \geq \frac{6n}{2} = 3n$.

Thus $\text{cr}(G) \geq m - (3n - 6) \geq 6$.



Critical Graphs and Proof of the Prop.

Def. G is a **critical graph** iff $\forall e \in E(G): \chi(G - e) < \chi(G)$.

Obs. Every G contains a critical subgraph H with $\chi(H) = \chi(G)$.

Obs. If G is r -critical, then $\delta(G) \geq r - 1$.

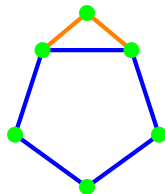
Prop. If $\chi(G) = 7$, then $\text{cr}(G) \geq 7$.

Pf. Assume G is 7-critical and $K_7 \not\subseteq G$.

Since $\chi(G) = 7$, $\delta(G) \geq 6$; so $m \geq \frac{6n}{2} = 3n$.

Thus $\text{cr}(G) \geq m - (3n - 6) \geq 6$.

Thm. (Brooks' Theorem) If G is connected and not a complete graph or odd cycle, then $\chi(G) \leq \Delta(G)$.



Critical Graphs and Proof of the Prop.

Def. G is a **critical graph** iff $\forall e \in E(G): \chi(G - e) < \chi(G)$.

Obs. Every G contains a critical subgraph H with $\chi(H) = \chi(G)$.

Obs. If G is r -critical, then $\delta(G) \geq r - 1$.

Prop. If $\chi(G) = 7$, then $\text{cr}(G) \geq 7$.

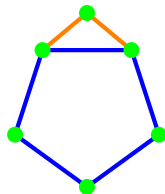
Pf. Assume G is 7-critical and $K_7 \not\subseteq G$.

Since $\chi(G) = 7$, $\delta(G) \geq 6$; so $m \geq \frac{6n}{2} = 3n$.

Thus $\text{cr}(G) \geq m - (3n - 6) \geq 6$.

Thm. (Brooks' Theorem) If G is connected and not a complete graph or odd cycle, then $\chi(G) \leq \Delta(G)$.

Thus, $\Delta(G) \geq 7$.



Critical Graphs and Proof of the Prop.

Def. G is a **critical graph** iff $\forall e \in E(G): \chi(G - e) < \chi(G)$.

Obs. Every G contains a critical subgraph H with $\chi(H) = \chi(G)$.

Obs. If G is r -critical, then $\delta(G) \geq r - 1$.

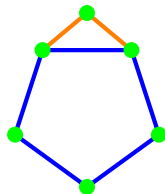
Prop. If $\chi(G) = 7$, then $\text{cr}(G) \geq 7$.

Pf. Assume G is 7-critical and $K_7 \not\subseteq G$.
Since $\chi(G) = 7$, $\delta(G) \geq 6$; so $m \geq \frac{6n}{2} = 3n$.
Thus $\text{cr}(G) \geq m - (3n - 6) \geq 6$.

Thm. (Brooks' Theorem) If G is connected and not a complete graph or odd cycle, then $\chi(G) \leq \Delta(G)$.

Thus, $\Delta(G) \geq 7$.

Hence $m \geq 3n + 1$ and $\text{cr}(G) \geq m - (3n - 6) \geq 7$. ■



$\chi(G)$ and generalizations of planarity

4 Color Theorem: Every planar graph is 4-colorable.

$\chi(G)$ and generalizations of planarity

4 Color Theorem: Every planar graph is 4-colorable.

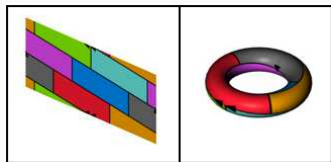
Relaxations of Planarity

$\chi(G)$ and generalizations of planarity

4 Color Theorem: Every planar graph is 4-colorable.

Relaxations of Planarity

Def. Genus of a graph G , $g(G)$:
min number of handles we must
add to the plane to embed G ,
e.g., $g(K_7) = 1$.

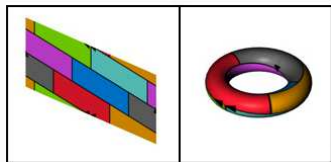


$\chi(G)$ and generalizations of planarity

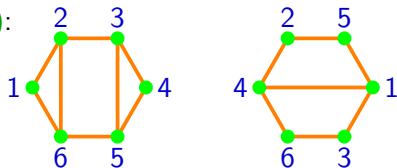
4 Color Theorem: Every planar graph is 4-colorable.

Relaxations of Planarity

Def. Genus of a graph G , $g(G)$:
min number of handles we must
add to the plane to embed G ,
e.g., $g(K_7) = 1$.



Def. Thickness of a graph G , $\tau(G)$:
min k such that $E(G)$ has a
partition into k planar graphs,
e.g., $\tau(K_6) = 2$.

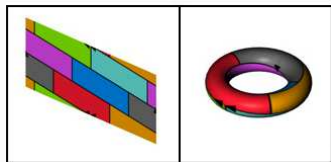


$\chi(G)$ and generalizations of planarity

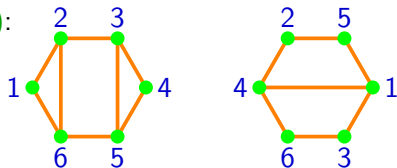
4 Color Theorem: Every planar graph is 4-colorable.

Relaxations of Planarity

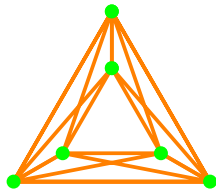
Def. Genus of a graph G , $g(G)$:
min number of handles we must
add to the plane to embed G ,
e.g., $g(K_7) = 1$.



Def. Thickness of a graph G , $\tau(G)$:
min k such that $E(G)$ has a
partition into k planar graphs,
e.g., $\tau(K_6) = 2$.



Def. Crossing number, $cr(G)$;
e.g., $cr(K_6) = 3$.

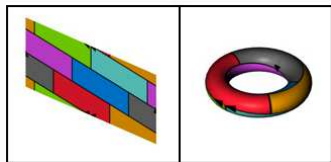


$\chi(G)$ and generalizations of planarity

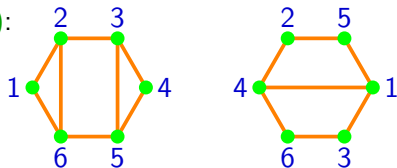
4 Color Theorem: Every planar graph is 4-colorable.

Relaxations of Planarity

Def. Genus of a graph G , $g(G)$:
min number of handles we must
add to the plane to embed G ,
e.g., $g(K_7) = 1$.

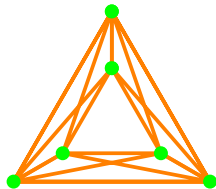


Def. Thickness of a graph G , $\tau(G)$:
min k such that $E(G)$ has a
partition into k planar graphs,
e.g., $\tau(K_6) = 2$.



Def. Crossing number, $cr(G)$;
e.g., $cr(K_6) = 3$.

Bound $\chi(G)$ in $g(G)$, $\tau(G)$, or $cr(G)$?

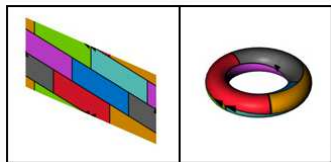


$\chi(G)$ and generalizations of planarity

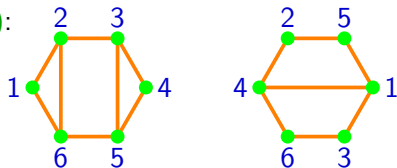
4 Color Theorem: Every planar graph is 4-colorable.

Relaxations of Planarity

Def. Genus of a graph G , $g(G)$:
min number of handles we must
add to the plane to embed G ,
e.g., $g(K_7) = 1$.

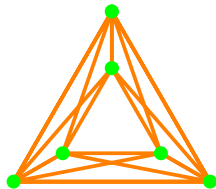


Def. Thickness of a graph G , $\tau(G)$:
min k such that $E(G)$ has a
partition into k planar graphs,
e.g., $\tau(K_6) = 2$.



Def. Crossing number, $cr(G)$;
e.g., $cr(K_6) = 3$.

Bound $\chi(G)$ in $g(G)$, $\tau(G)$, or $cr(G)$?
If so, what are the extremal graphs?



Easier proofs of harder results

Outline of Meta-proof

Easier proofs of harder results

Outline of Meta-proof

0. Consider only critical G .

Easier proofs of harder results

Outline of Meta-proof

0. Consider only critical G .
1. Prove lower bound on m .

Easier proofs of harder results

Outline of Meta-proof

0. Consider only critical G .
1. Prove lower bound on m . (e.g. $m \geq \frac{r-1}{2}n + 1$)

Easier proofs of harder results

Outline of Meta-proof

0. Consider only critical G .
1. Prove lower bound on m . (e.g. $m \geq \frac{r-1}{2}n + 1$)
2. Prove lower bound on $\text{cr}(G)$ in m .

Easier proofs of harder results

Outline of Meta-proof

0. Consider only critical G .
1. Prove lower bound on m . (e.g. $m \geq \frac{r-1}{2}n + 1$)
2. Prove lower bound on $\text{cr}(G)$ in m . (e.g. $\text{cr}(G) \geq m - (3n - 6)$)

Easier proofs of harder results

Outline of Meta-proof

0. Consider only critical G .
1. Prove lower bound on m . (e.g. $m \geq \frac{r-1}{2}n + 1$)
2. Prove lower bound on $\text{cr}(G)$ in m . (e.g. $\text{cr}(G) \geq m - (3n - 6)$)

Idea. Improvements in 1. or 2. should help us prove more cases of Albertson's Conjecture.

Easier proofs of harder results

Outline of Meta-proof

0. Consider only critical G .
1. Prove lower bound on m . (e.g. $m \geq \frac{r-1}{2}n + 1$)
2. Prove lower bound on $\text{cr}(G)$ in m . (e.g. $\text{cr}(G) \geq m - (3n - 6)$)

Idea. Improvements in 1. or 2. should help us prove more cases of Albertson's Conjecture.

Thm. [Dirac '52] If G is r -critical and $G \neq K_r$, then

$$m \geq \frac{r-1}{2}n + \frac{r-3}{2}.$$

Easier proofs of harder results

Outline of Meta-proof

0. Consider only critical G .
1. Prove lower bound on m . (e.g. $m \geq \frac{r-1}{2}n + 1$)
2. Prove lower bound on $\text{cr}(G)$ in m . (e.g. $\text{cr}(G) \geq m - (3n - 6)$)

Idea. Improvements in 1. or 2. should help us prove more cases of Albertson's Conjecture.

Thm. [Dirac '52] If G is r -critical and $G \neq K_r$, then

$$m \geq \frac{r-1}{2}n + \frac{r-3}{2}.$$

Thm. [Kostochka-Stiebitz '96] If G is r -critical and $G \neq K_r$ and $n \neq 2r - 1$, then

$$m \geq \frac{r-1}{2}n + r - 3.$$

Proving Albertson's Conjecture (for lots more cases)

Crossing Lemma [Leighton; Ajtai et. al. '82] If $m \geq 4n$, then

$$\text{cr}(G) \geq \frac{1}{64} \frac{m^3}{n^2}.$$

Proving Albertson's Conjecture (for lots more cases)

Crossing Lemma [Pach et. al. '06] If $m \geq (103/16)n$, then

$$\text{cr}(G) \geq \frac{1}{31.1} \frac{m^3}{n^2}.$$

Proving Albertson's Conjecture (for lots more cases)

Crossing Lemma [Pach et. al. '06] If $m \geq (103/16)n$, then

$$\text{cr}(G) \geq \frac{1}{31.1} \frac{m^3}{n^2}.$$

Thm. [Pach et. al. '06]

$$\text{cr}(G) \geq (7/3)m - (25/3)(n - 2)$$

$$\text{cr}(G) \geq 3m - (35/3)(n - 2)$$

$$\text{cr}(G) \geq 4m - (103/6)(n - 2)$$

Proving Albertson's Conjecture (for lots more cases)

Crossing Lemma [Pach et. al. '06] If $m \geq (103/16)n$, then

$$\text{cr}(G) \geq \frac{1}{31.1} \frac{m^3}{n^2}.$$

Thm. [Pach et. al. '06]

$$\text{cr}(G) \geq (7/3)m - (25/3)(n - 2)$$

$$\text{cr}(G) \geq 3m - (35/3)(n - 2)$$

$$\text{cr}(G) \geq 4m - (103/6)(n - 2)$$

Prop. Albertson's Conjecture for $r = 9$. (Recall $\text{cr}(K_9) = 36$.)

Proving Albertson's Conjecture (for lots more cases)

Crossing Lemma [Pach et. al. '06] If $m \geq (103/16)n$, then

$$\text{cr}(G) \geq \frac{1}{31.1} \frac{m^3}{n^2}.$$

Thm. [Pach et. al. '06]

$$\text{cr}(G) \geq (7/3)m - (25/3)(n - 2)$$

$$\text{cr}(G) \geq 3m - (35/3)(n - 2)$$

$$\text{cr}(G) \geq 4m - (103/6)(n - 2)$$

Prop. Albertson's Conjecture for $r = 9$. (Recall $\text{cr}(K_9) = 36$.)

Pf. Assume G is 9-critical and $G \neq K_9$. Note $n \geq 10$.

Proving Albertson's Conjecture (for lots more cases)

Crossing Lemma [Pach et. al. '06] If $m \geq (103/16)n$, then

$$\text{cr}(G) \geq \frac{1}{31.1} \frac{m^3}{n^2}.$$

Thm. [Pach et. al. '06]

$$\text{cr}(G) \geq (7/3)m - (25/3)(n - 2)$$

$$\text{cr}(G) \geq 3m - (35/3)(n - 2)$$

$$\text{cr}(G) \geq 4m - (103/6)(n - 2)$$

Prop. Albertson's Conjecture for $r = 9$. (Recall $\text{cr}(K_9) = 36$.)

Pf. Assume G is 9-critical and $G \neq K_9$. Note $n \geq 10$.

If $n \neq 17$, then Kostochka-Stiebitz bound gives $m \geq 4n + 6$,
so $\text{cr}(G) \geq (7/3)m - (25/3)(n - 2) \geq n + (92/3) > 40$.

Proving Albertson's Conjecture (for lots more cases)

Crossing Lemma [Pach et. al. '06] If $m \geq (103/16)n$, then

$$\text{cr}(G) \geq \frac{1}{31.1} \frac{m^3}{n^2}.$$

Thm. [Pach et. al. '06]

$$\text{cr}(G) \geq (7/3)m - (25/3)(n - 2)$$

$$\text{cr}(G) \geq 3m - (35/3)(n - 2)$$

$$\text{cr}(G) \geq 4m - (103/6)(n - 2)$$

Prop. Albertson's Conjecture for $r = 9$. (Recall $\text{cr}(K_9) = 36$.)

Pf. Assume G is 9-critical and $G \neq K_9$. Note $n \geq 10$.

If $n \neq 17$, then Kostochka-Stiebitz bound gives $m \geq 4n + 6$,
so $\text{cr}(G) \geq (7/3)m - (25/3)(n - 2) \geq n + (92/3) > 40$.

If $n = 17$, then Dirac's bound gives $m \geq 4n + 3$,
so $\text{cr}(G) \geq (7/3)m - (25/3)(n - 2) \geq 122/3 > 40$. ■

Proving Albertson's Conjecture (for lots more cases)

Crossing Lemma [Pach et. al. '06] If $m \geq (103/16)n$, then

$$\text{cr}(G) \geq \frac{1}{31.1} \frac{m^3}{n^2}.$$

Thm. [Pach et. al. '06]

$$\text{cr}(G) \geq (7/3)m - (25/3)(n - 2)$$

$$\text{cr}(G) \geq 3m - (35/3)(n - 2)$$

$$\text{cr}(G) \geq 4m - (103/6)(n - 2)$$

Prop. Albertson's Conjecture for $r = 9$. (Recall $\text{cr}(K_9) = 36$.)

Pf. Assume G is 9-critical and $G \neq K_9$. Note $n \geq 10$.

If $n \neq 17$, then Kostochka-Stiebitz bound gives $m \geq 4n + 6$,
so $\text{cr}(G) \geq (7/3)m - (25/3)(n - 2) \geq n + (92/3) > 40$.

If $n = 17$, then Dirac's bound gives $m \geq 4n + 3$,
so $\text{cr}(G) \geq (7/3)m - (25/3)(n - 2) \geq 122/3 > 40$. ■

Thm. Albertson's Conjecture is true for $r \leq 12$.