Oligomorphic permutation groups: growth rates and algebras

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The definition

Let G be a permutation group on an infinite set Ω. Then G has a natural induced action on the set of all n-tuples of elements of Ω, or on the set of n-tuples of distinct elements of Ω, or on the set of n-element subsets of Ω. It is easy to see that if there are only finitely many orbits on one of these sets, then the same is true for the others.

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We denote the number of orbits on all n-tuples, resp. n-tuples of distinct elements, n-sets, by $F_n^*(G)$, $F_n(G)$, $f_n(G)$ respectively.

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- $f_n(A) = 1$;
- $F_n(A) = n!$;
- $F_n^*(A)$ is the number of preorders of an n-set.

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For A^2 acting on $\mathbb{Q}^2$, $f_n(A^2)$ is the number of zero-one matrices (of unspecified size) with n ones and no rows or columns of zeros.

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Let $G = S_2 \operatorname{Wr} A$, where S_2 is the symmetric group of degree 2. Then $f_n(G)$ is the nth Fibonacci number.

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If $G = \mathrm{Aut}(R)$, then $F_n(G)$ and $f_n(G)$ are the numbers of labelled and unlabelled graphs on n vertices.

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What other structures can be specified by countability and first-order axioms? Such structures are called countably categorical.

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In fact, more is true: the types over the theory of M are all realised in M, and the sets of n-tuples which realise the n-types are precisely the orbits of $\mathrm{Aut}(M)$ on M^n.

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- if G is highly homogeneous (that is, if $f_n(G) = 1$ for all n), then either there is a linear or circular order on Ω preserved or reversed by G, or G is highly transitive (that is, $F_n(G) = 1$ for all n).

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- There is no upper bound on the growth rate of $(f_n(G))$.

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and so on.
I do not know how to prove any of these things; and I do not know how to formulate a general conjecture.

A Ramsey-type theorem

Theorem

Let X be an infinite set, and suppose that the n-element subsets of Ω are coloured with r different colours (all of which are used). Then there is an ordering $(c_1, \ldots, c_r)$ of the colours, and infinite subsets $Y_1, \ldots, Y_r$ of X, such that, for $i = 1, \ldots, r$, the set Y_i contains an n-set of colour c_i but none of colour c_j for $j > i$.

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There is a finite version of the theorem, and so there are corresponding 'Ramsey numbers'. But very little is known about them!

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Proof.

Let $r = f_n(G)$, and colour the n-subsets with r colours according to the orbits. Then by the Theorem, there exists an $(n+1)$-set containing a set of colour c_i but none of colour c_j for $j > i$. These $(n+1)$-sets all lie in different orbits; so $f_{n+1}(G) \geq r$. □

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There is also an algebraic proof of this corollary. We'll discuss this later.

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Let $\binom{\Omega}{n}$ denote the set of n-subsets of Ω, and V_n the vector space of functions from $\binom{\Omega}{n}$ to $\mathbb{C}$.

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We make $\mathcal{A} = \bigoplus_{n \geq 0} V_n$ into an algebra by defining, for $f \in V_n$, $g \in V_m$, the product $fg \in V_{n+m}$ by

$$(fg)(K) = \sum_{M \in \binom{K}{m}} f(M) g(K \setminus M)$$

for $K \in \binom{\Omega}{m+n}$, and extending linearly.

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$\mathcal{A}$ is a commutative and associative graded algebra over $\mathbb{C}$, sometimes referred to as the reduced incidence algebra of finite subsets of Ω.

A graded algebra, 2

Now let G be a permutation group on Ω, and let V_n^G denote the set of fixed points of G in V_n. Put

$$\mathcal{A}[G] = \bigoplus_{n \geq 0} V_n^G,$$

a graded subalgebra of $\mathcal{A}$.

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Note that it is not usually finitely generated since the growth of $(f_n(G))$ is polynomial only in special cases.

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This theorem gives another proof of the monotonicity of $(f_n(G))$. For multiplication by e is a monomorphism from V_n^G to V_{n+1}^G, and so $f_{n+1}(G) = \dim v_{n+1}^G \geq \dim V_n^G = f_n(G)$.

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The converse, a long-standing conjecture, has recently been proved by Maurice Pouzet:

Theorem

If G has no finite orbits on Ω, then $\mathcal{A}[G]$ is an integral domain.

Consequences

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It seems very likely that better understanding of the algebra $\mathcal{A}[G]$ would have further implications for growth rate.

Brief sketch of the proof

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Pouzet shows that, if $f \in V_m$ and $g \in V_n$ satisfy $fg = 0$, then the transversality of $\mathrm{supp}(f) \cup \mathrm{supp}(g)$ is finite, and is bounded by a function of m and n. (Here $\mathrm{supp}(f)$ denotes the support of f.)

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These two results clearly conflict with each other.

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The proof of this makes it clear that it is another kind of 'Ramsey theorem'. If $\tau(m,n)$ denotes the smallest t such that the transversality is at most t, then we have the interesting problem of finding $\tau(m,n)$. Pouzet shows that $\tau(m,n) \geq (m+1)(n+1) - 1$. On the other hand, the upper bounds coming from his proof are really astronomical!