

Oligomorphic permutation groups: growth rates and algebras

Peter J. Cameron



p.j.cameron@qmul.ac.uk

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The definition

Let G be a permutation group on an infinite set Ω . Then G has a natural induced action on the set of all n -tuples of elements of Ω , or on the set of n -tuples of distinct elements of Ω , or on the set of n -element subsets of Ω . It is easy to see that if there are only finitely many orbits on one of these sets, then the same is true for the others.

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We say that G is **oligomorphic** if it has only finitely many orbits on Ω^n for all natural numbers n .

We denote the number of orbits on all n -tuples, resp. n -tuples of distinct elements, n -sets, by $F_n^*(G)$, $F_n(G)$, $f_n(G)$ respectively.

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- ▶ $f_n(A) = 1$;
- ▶ $F_n(A) = n!$;
- ▶ $F_n^*(A)$ is the number of **preorders** of an n -set.

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Consider S^r acting on Ω^r . Then $F_n^*(S^r) = B(n)^r$. From this we can find $F_n(S^r)$ by inversion:

$$F_n(G) = \sum_{k=1}^n s(n, k) F_k^*(G)$$

for any oligomorphic group G , where $s(n, k)$ is the signed **Stirling number** of the second kind.

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For A^2 acting on \mathbb{Q}^2 , $f_n(A^2)$ is the number of zero-one matrices (of unspecified size) with n ones and no rows or columns of zeros.

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Let $G = S \text{ Wr } S$, the wreath product of two copies of S . Then $F_n(G) = B(n)$ and $f_n(G) = p(n)$, the number of partitions of n .

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Let $G = S_2 \text{ Wr } A$, where S_2 is the symmetric group of degree 2. Then $f_n(G)$ is the n th **Fibonacci number**.

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If $G = \text{Aut}(R)$, then $F_n(G)$ and $f_n(G)$ are the numbers of labelled and unlabelled graphs on n vertices.

Connection with model theory, 1

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What other structures can be specified by countability and first-order axioms? Such structures are called **countably categorical**.

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In fact, more is true: the **types** over the theory of M are all realised in M , and the sets of n -tuples which realise the n -types are precisely the orbits of $\text{Aut}(M)$ on M^n .

Growth of $(f_n(G))$, 1

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- ▶ if G is **highly homogeneous** (that is, if $f_n(G) = 1$ for all n), then either there is a linear or circular order on Ω preserved or reversed by G , or G is **highly transitive** (that is, $F_n(G) = 1$ for all n).

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- ▶ There is no upper bound on the growth rate of $(f_n(G))$.

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- ▶ for exponential growth, $\lim(\log f_n(G) / n)$ should exist;

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I do not know how to prove any of these things; and I do not know how to formulate a general conjecture.

A Ramsey-type theorem

Theorem

Let X be an infinite set, and suppose that the n -element subsets of Ω are coloured with r different colours (all of which are used). Then there is an ordering (c_1, \dots, c_r) of the colours, and infinite subsets Y_1, \dots, Y_r of X , such that, for $i = 1, \dots, r$, the set Y_i contains an n -set of colour c_i but none of colour c_j for $j > i$.

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There is a finite version of the theorem, and so there are corresponding 'Ramsey numbers'. But very little is known about them!

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Proof.

Let $r = f_n(G)$, and colour the n -subsets with r colours according to the orbits. Then by the Theorem, there exists an $(n + 1)$ -set containing a set of colour c_i but none of colour c_j for $j > i$. These $(n + 1)$ -sets all lie in different orbits; so $f_{n+1}(G) \geq r$. \square

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There is also an algebraic proof of this corollary. We'll discuss this later.

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We make $\mathcal{A} = \bigoplus_{n \geq 0} V_n$ into an algebra by defining, for $f \in V_n$, $g \in V_m$, the product $fg \in V_{n+m}$ by

$$(fg)(K) = \sum_{M \in \binom{K}{m}} f(M)g(K \setminus M)$$

for $K \in \binom{\Omega}{m+n}$, and extending linearly.

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\mathcal{A} is a commutative and associative graded algebra over \mathbb{C} , sometimes referred to as the **reduced incidence algebra** of finite subsets of Ω .

A graded algebra, 2

Now let G be a permutation group on Ω , and let V_n^G denote the set of fixed points of G in V_n . Put

$$\mathcal{A}[G] = \bigoplus_{n \geq 0} V_n^G,$$

a graded subalgebra of \mathcal{A} .

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Note that it is not usually finitely generated since the growth of $(f_n(G))$ is polynomial only in special cases.

A non-zero-divisor

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This theorem gives another proof of the monotonicity of $(f_n(G))$. For multiplication by e is a monomorphism from V_n^G to V_{n+1}^G , and so $f_{n+1}(G) = \dim v_{n+1}^G \geq \dim V_n^G = f_n(G)$.

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The converse, a long-standing conjecture, has recently been proved by Maurice Pouzet:

Theorem

If G has no finite orbits on Ω , then $\mathcal{A}[G]$ is an integral domain.

Consequences

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It seems very likely that better understanding of the algebra $\mathcal{A}[G]$ would have further implications for growth rate.

Brief sketch of the proof

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Pouzet shows that, if $f \in V_m$ and $g \in V_n$ satisfy $fg = 0$, then the transversality of $\text{supp}(f) \cup \text{supp}(g)$ is finite, and is bounded by a function of m and n . (Here $\text{supp}(f)$ denotes the support of f .)

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These two results clearly conflict with each other.

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The proof of this makes it clear that it is another kind of 'Ramsey theorem'. If $\tau(m, n)$ denotes the smallest t such that the transversality is at most t , then we have the interesting problem of finding $\tau(m, n)$. Pouzet shows that $\tau(m, n) \geq (m + 1)(n + 1) - 1$. On the other hand, the upper bounds coming from his proof are really astronomical!