

Leaf Powers: Recent Results and Open Problems

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Contents

1. Basic notions
2. Leaf powers vs. fixed tolerance NeST graphs
3. Inclusions of k -leaf power classes
4. On 3-, 4-, and 5-leaf powers
5. On (k,l) -leaf powers
6. On simplicial powers

Leaf Powers

[Nishimura, Ragde, Thilikos, On graph powers for leaf-labeled trees, *J. Algorithms* 2002]:

A finite undirected graph $G = (V, E)$ is a *k-leaf power* if there is a tree $T = (U, F)$ whose set of leaves is V such that for all $x, y \in V$

$$xy \in E \Leftrightarrow \text{dist}_T(x, y) \leq k.$$

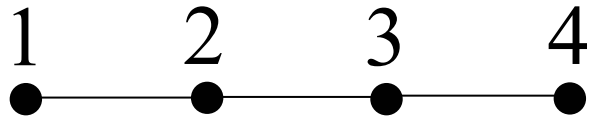
T is called a *k-leaf root of G*.

Leaf Powers

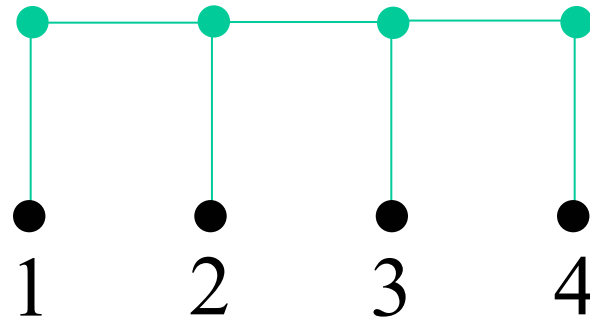
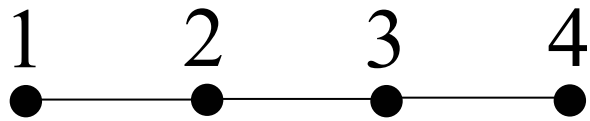
A graph is a *leaf power* if it is a k -leaf power for some $k \geq 2$.

Obviously, the 2-leaf powers are exactly the disjoint unions of cliques.

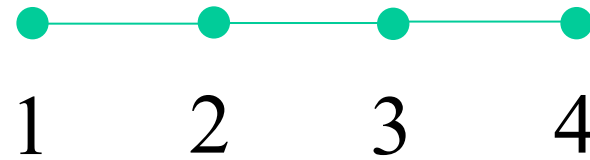
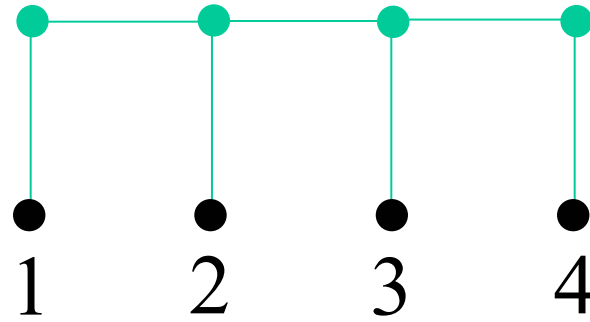
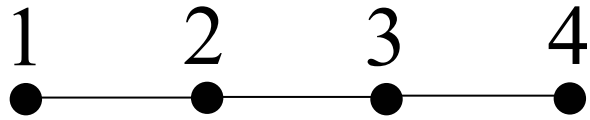
Leaf Powers



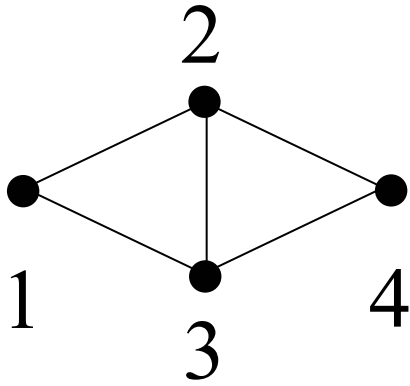
Leaf Powers



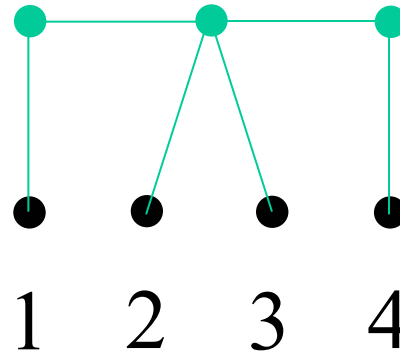
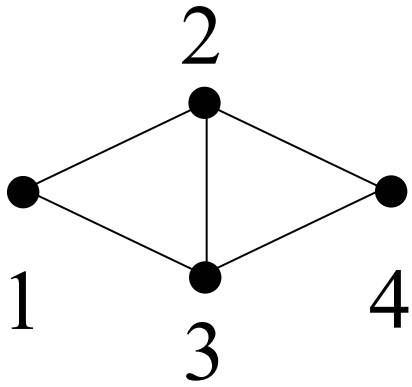
Leaf Powers



Leaf Powers

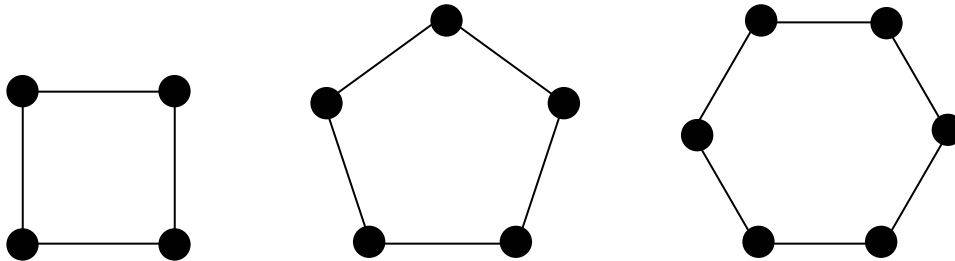


Leaf Powers



Chordal Graphs

Graph G is *chordal* if it contains no chordless cycles of length at least four.



Graph Powers

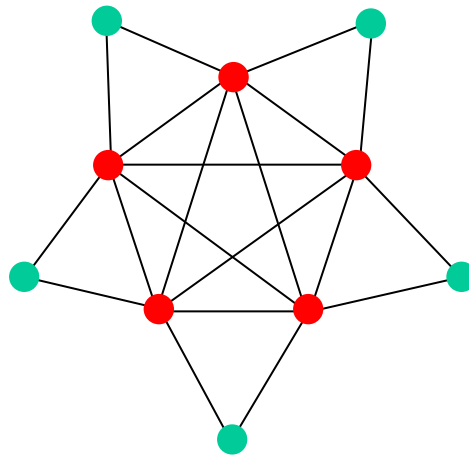
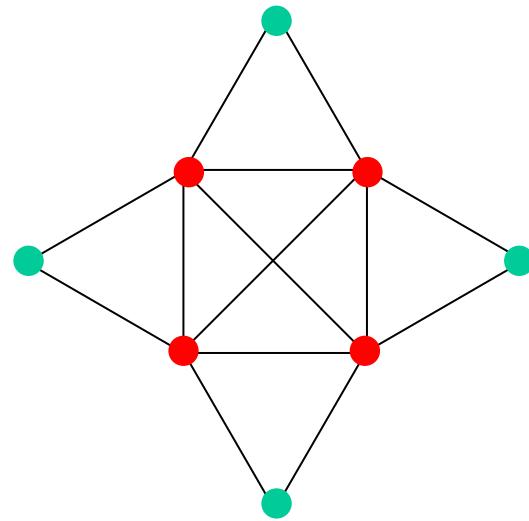
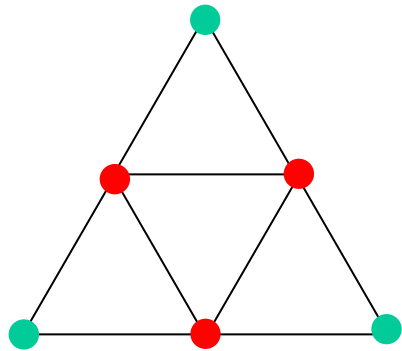
For graph $G = (V, E)$, let $G^k = (V, E^k)$ with

$$xy \in E^k \Leftrightarrow \text{dist}_G(x, y) \leq k$$

denote the *k -th power of G* .

Fact. Any k -leaf power is an induced subgraph of the k -th power of a tree. Any induced subgraph of a k -leaf power is a k -leaf power.

Fact. Powers of trees are chordal.



Leaf powers are strongly chordal

A graph is *strongly chordal* if it is chordal and sun-free. Trees are strongly chordal.

Theorem [Lubiw, 1982; Dahlhaus, Duchet, 1987; Raychaudhuri, 1992] For every $k \geq 2$:

G strongly chordal $\Rightarrow G^k$ strongly chordal.

Corollary For every $k \geq 2$, k -leaf powers are strongly chordal.

Neighborhood Subtree Tolerance Graphs

[Bibelnieks, Dearing, 1993, Neighborhood subtree tolerance graphs]:

NeST graphs are weakly chordal.

[Hayward, Kearney, 1993; Hayward, Kearney, Malton, 2002]:

NeST graphs are weakly chordal but not vice versa.

Neighborhood Subtree Tolerance Graphs

Theorem [Hayward, Kearney, 1993; Hayward, Kearney, Malton, 2002]:

A graph $G = (V, E)$ is a NeST graph with **fixed tolerance** \Leftrightarrow

\exists positive constant $k > 0$ and an undirected weighted tree $T = (N, A, w)$ with $V \subseteq N$ and positive weights $w: A \rightarrow R$ such that

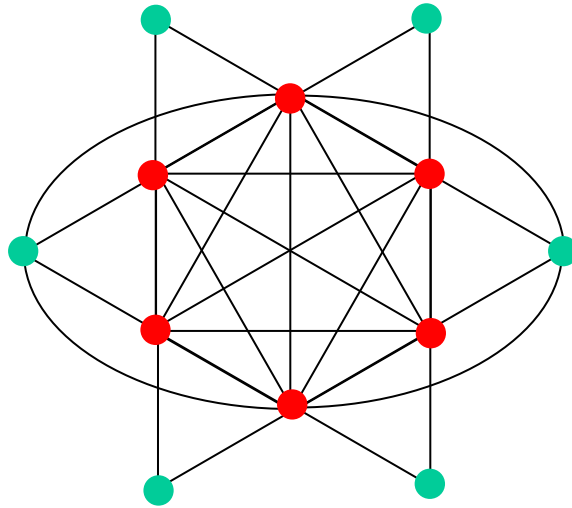
$$\forall u, v \in V (uv \in E \Leftrightarrow d_T(u, v) \leq k)$$

Neighborhood Subtree Tolerance Graphs

Fact. Fixed Tolerance NeST graphs are strongly chordal but not vice versa.

(proof is based on [Broin, Lowe, A dynamic programming algorithm for covering problems with (greedy) totally balanced constraint matrices, 1986])

No Fixed Tolerance NeST graph



Neighborhood Subtree Tolerance Graphs vs. Leaf Powers

Theorem [B., Hundt, Mancini, Wagner, 2008]:

G is a fixed tolerance NeST graph \Leftrightarrow

G is a leaf power.

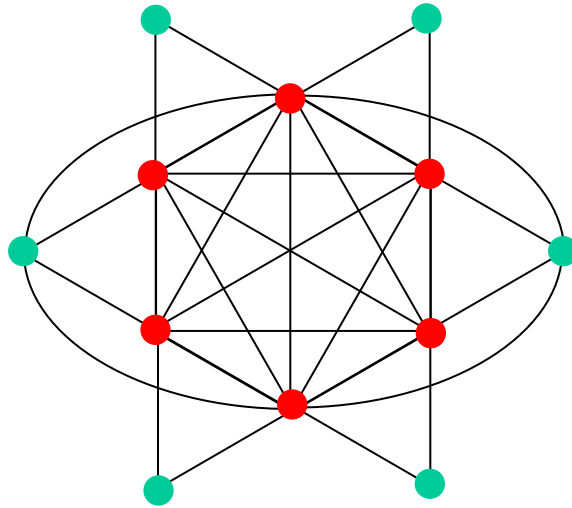
Corollary [B., Hundt, LATIN 2008]:

Interval graphs are leaf powers.

Theorem [B., Hundt, Mancini, Wagner, 2008]:

Rooted directed path graphs are leaf powers.

No Leaf Power



Inclusions

Let $L(k)$ denote the class of k -leaf powers.

Obviously, $L(k) \subset L(k+2)$ (by simply subdividing pendant edges in a k -leaf root).

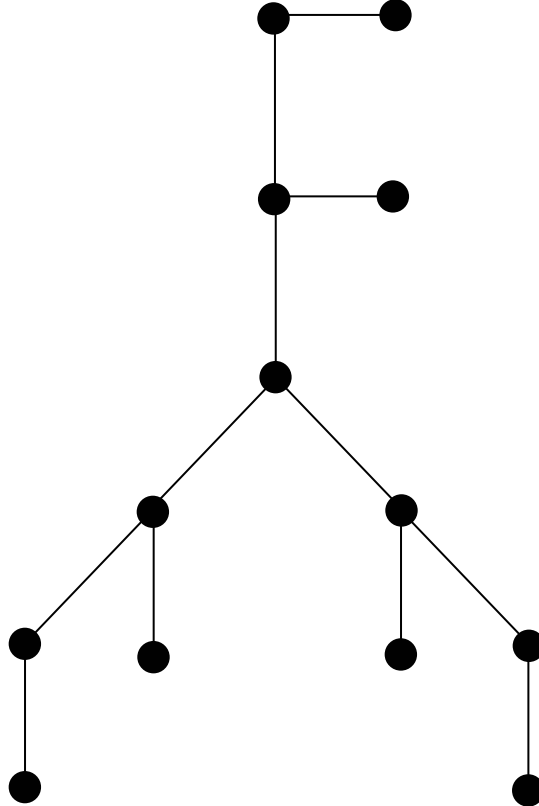
For a while, the following has been an open question:

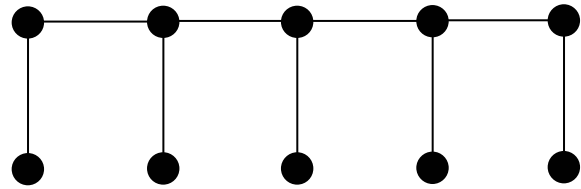
Is $L(k) \subset L(k+1)$?

Inclusions

[Fellows, Meister, Rosamond, Sritharan, Telle,
2007]:

$L(4) \not\subseteq L(5)$.





Inclusions

Theorem [Wagner, B. 2008] For any $2 \leq k < k'$
 $L(k) \not\subset L(k') \Leftrightarrow k' \leq 2k - 3$ and $k' - k$ is odd.

Examples

$L(4) \subset L(6)$ and $L(4) \subset L(7)$ but $L(4) \not\subset L(5)$.

$L(5) \subset L(7)$ and $L(5) \subset L(8)$ but $L(5) \not\subset L(6)$.

$L(6) \subset L(8)$ and $L(6) \subset L(11)$ but

$L(6) \not\subset L(7)$ and $L(6) \not\subset L(9)$.

Inclusions

Corollary

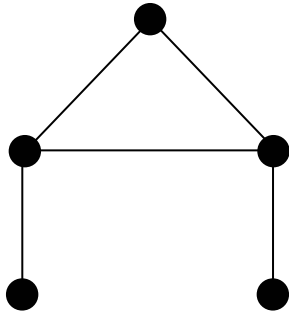
For all $k \geq 5$: $L(k) \not\subset L(k+1)$.

Corollary

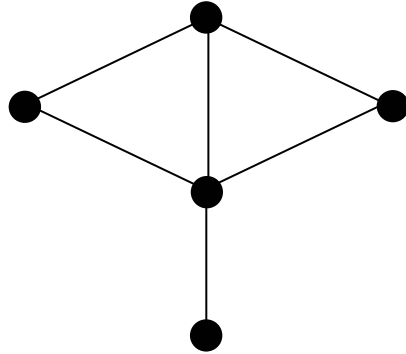
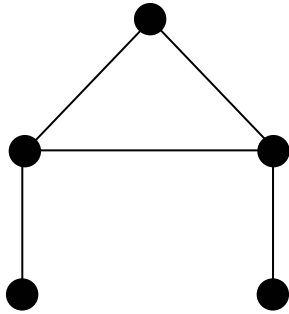
For $2 \leq k < k'$, the k -leaf power G is a k' -leaf power \Leftrightarrow

a k -leaf root T of G can be transformed to a k' -leaf root T' of G by subdividing edges of T .

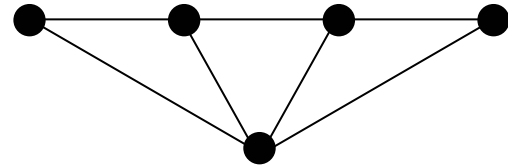
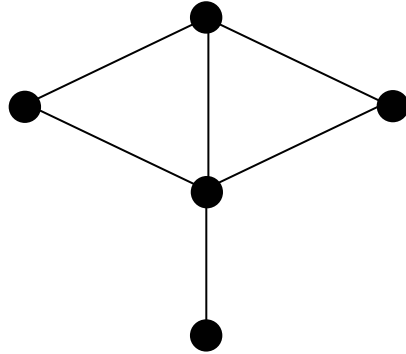
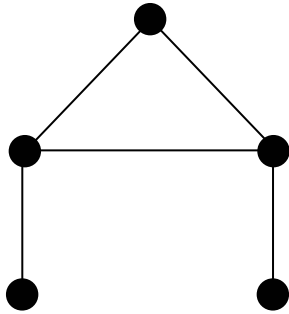
3-Leaf Powers



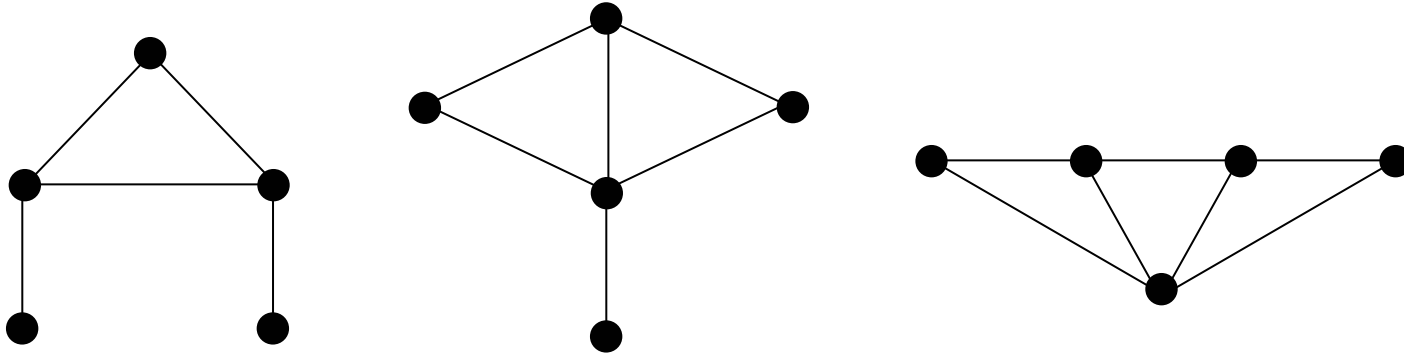
3-Leaf Powers



3-Leaf Powers



3-Leaf Powers



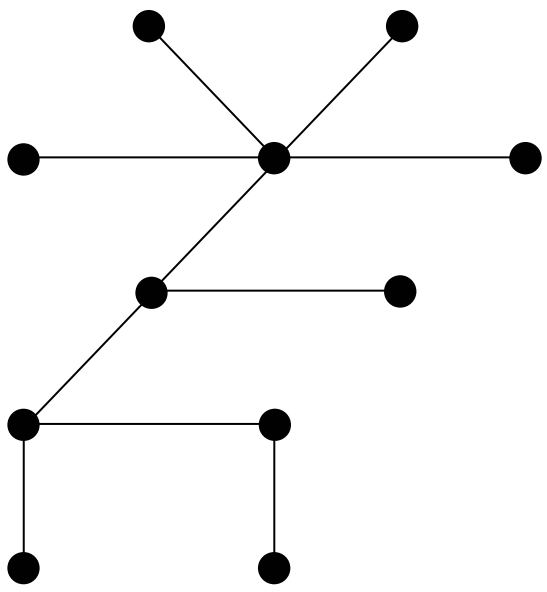
Theorem [Dom, Guo, Hüffner, Niedermeier, 2004]

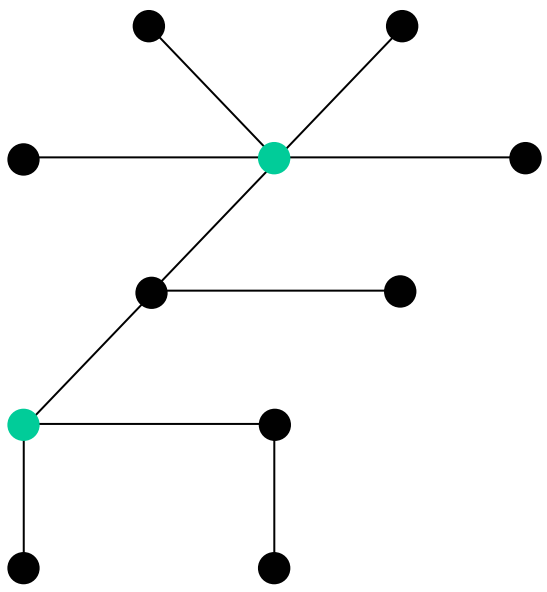
G is a 3-leaf power $\Leftrightarrow G$ is (bull, dart, gem)-free chordal \Leftrightarrow critical clique graph of G is a tree.

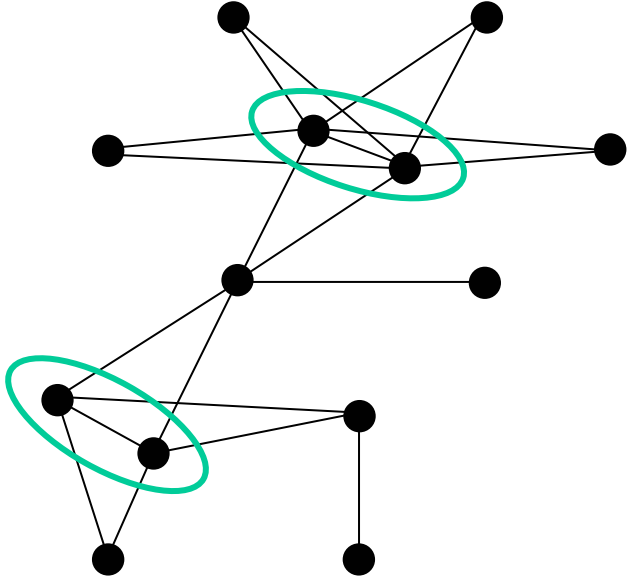
3-Leaf Powers

[B., Le, 2005; Rautenbach, 2004]

A connected graph G is a 3-leaf power \Leftrightarrow
 G results from substituting cliques into the
vertices of a tree.







3-Leaf Powers

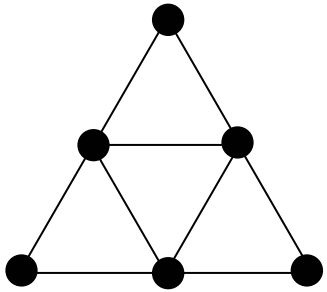
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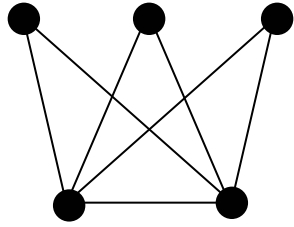
[B., Le 2005]

Linear time recognition for 3-leaf powers.

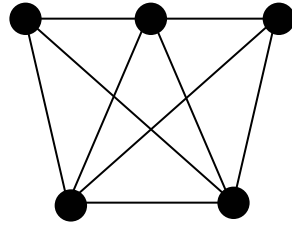
4-Leaf Powers



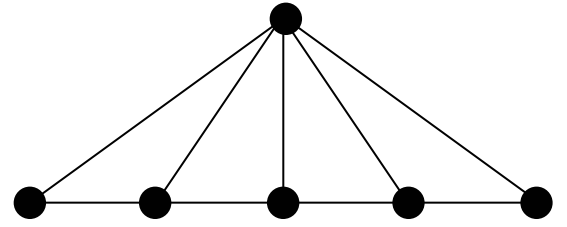
G_1



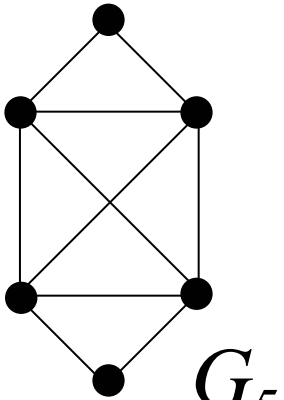
G_2



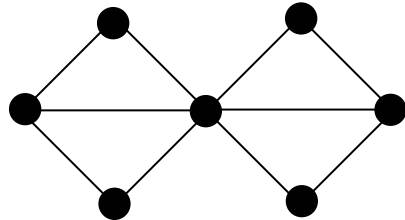
G_3



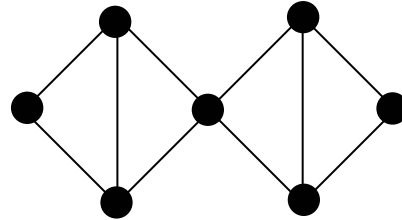
G_4



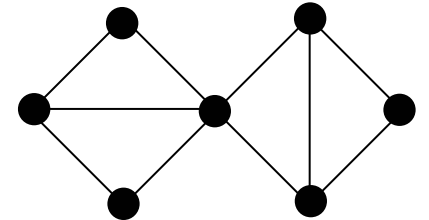
G_5



G_6



G_7



G_8

4-Leaf Powers

Theorem [Rautenbach, 2004]

A graph G without true twins is a 4-leaf power
 $\Leftrightarrow G$ is (G_1, \dots, G_8) -free chordal.

4-Leaf Powers

Theorem [B., Le, Sritharan, 2005]

For every graph G , the following conditions are equivalent:

- (i) G is a 2-connected basic 4-leaf power.
- (ii) G is the square of some tree.
- (iii) G is chordal, 2-connected and (G_1, \dots, G_5) -free.

4-Leaf Powers

Theorem [B., Le, Sritharan, 2005]

The following conditions are equivalent:

- (i) G is a basic 4-leaf power.
- (ii) Every block of G is the square of some tree, and for every non-disjoint pair of blocks, at least one of them is a clique.
- (iii) G is an induced subgraph of the square of some tree.
- (iv) G is (G_1, \dots, G_8) -free chordal.

Distance-Hereditary 5-Leaf Powers

Theorem [B., Le, Rautenbach, 2006]

For every distance-hereditary graph G , the following conditions are equivalent:

- (i) G is a basic 5-leaf power.
- (ii) G is (F_1, \dots, F_{34}) -free chordal.
- (iii) G fulfills “various“ block and gluing conditions ...

(k,l) -Leaf Powers

A finite undirected graph $G = (V, E)$ is a *(k,l) -leaf power* if there is a tree $T = (U, F)$ with leaf set V such that for all $x, y \in V$

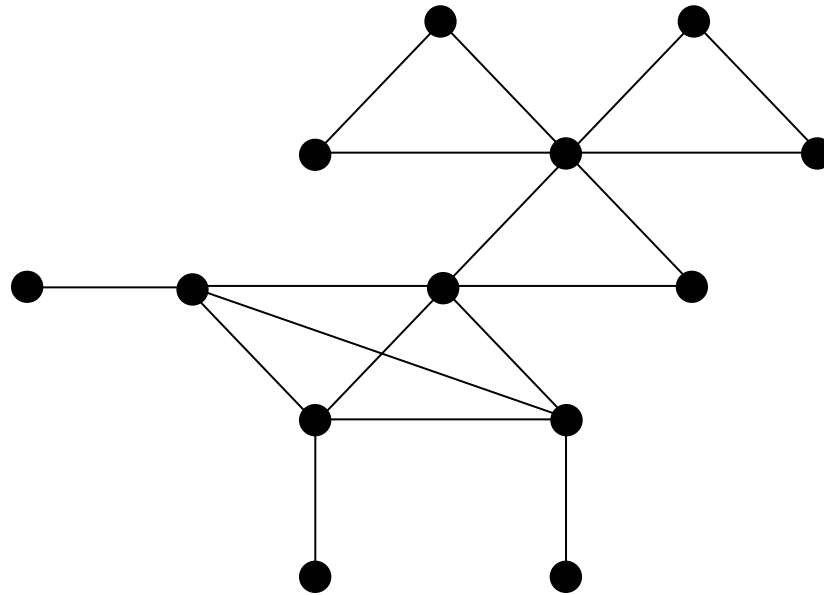
(i) $xy \in E \Rightarrow \text{dist}_T(x, y) \leq k$ and

(ii) $xy \notin E \Rightarrow \text{dist}_T(x, y) \geq l$.

Such a tree T is a *(k,l) -leaf root of G* .

Block Graphs

A (connected) graph G is a *block graph* if its blocks (i.e., 2-connected components) are cliques.



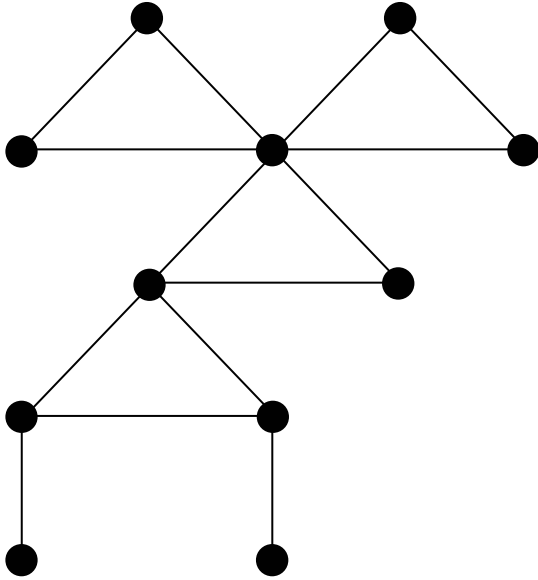
(4,6)-Leaf Powers

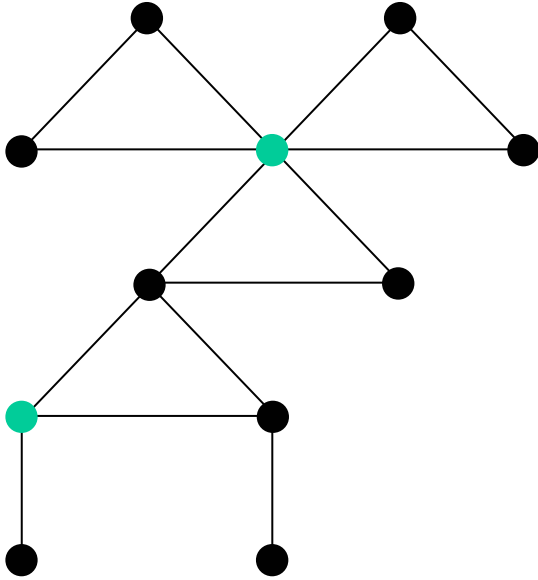
Theorem [B., Wagner, 2007]

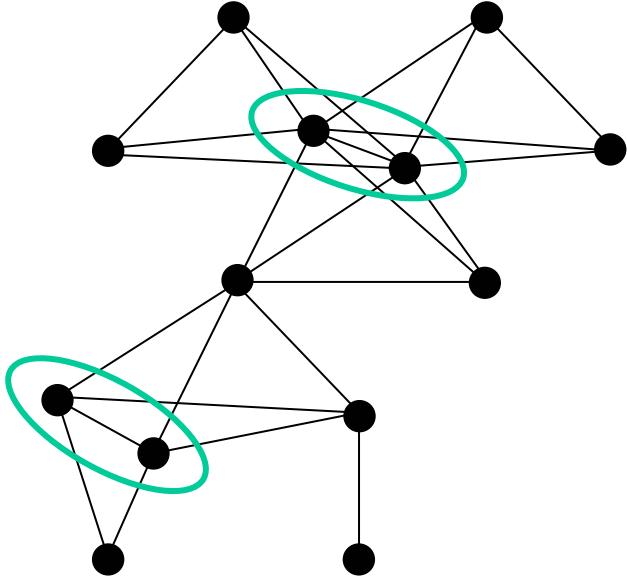
For connected graph G , the following are equivalent:

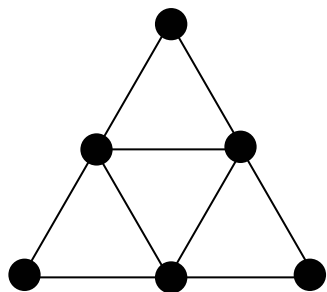
- (i) G is a (4,6)-leaf power.
- (ii) G is strictly chordal, i.e., (dart,gem)-free chordal.
- (iii) G results from a block graph by substituting cliques into its vertices.

(A paper by Kennedy, Lin and Yan 2006 shows that strictly chordal graphs are leaf powers.)

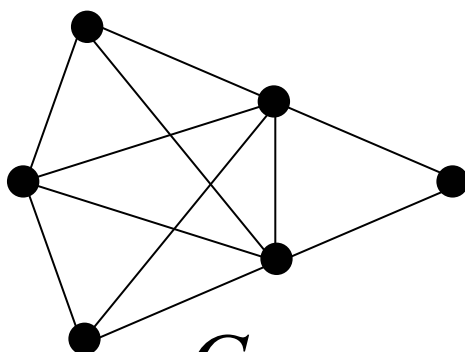




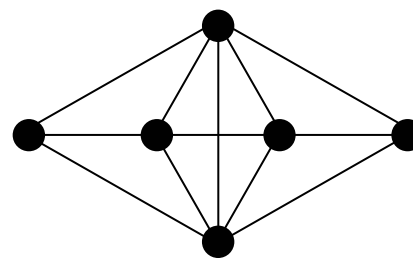




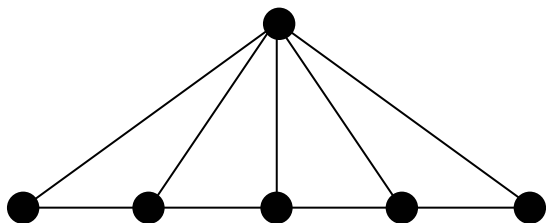
G_1



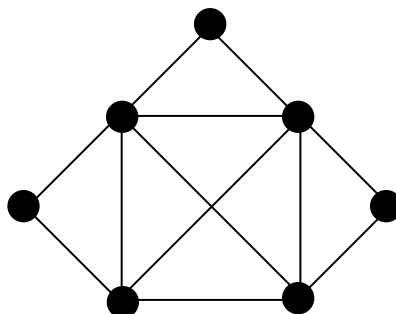
G_2



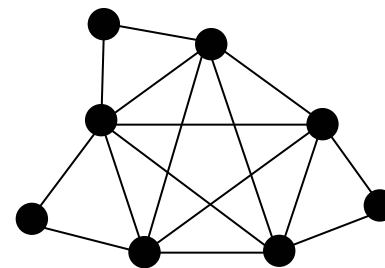
G_3



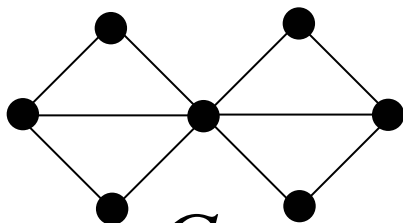
G_4



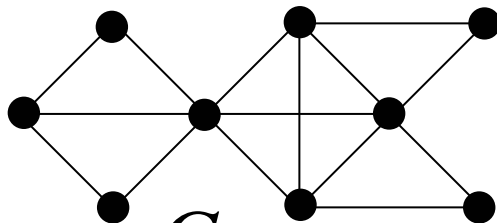
G_5



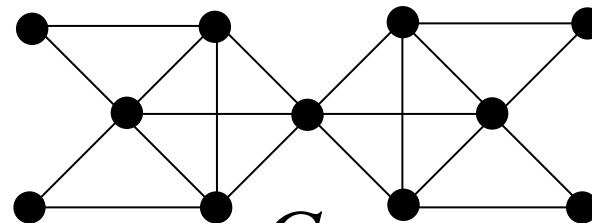
G_6



G_7



G_8



G_9

(6,8)-Leaf Powers

Theorem [B., Wagner, 2007]

The following conditions are equivalent:

- (i) G is a basic (6,8)-leaf power.
- (ii) G is an induced subgraph of the square of some block graph.
- (iii) G is (G_1, \dots, G_9) -free chordal.

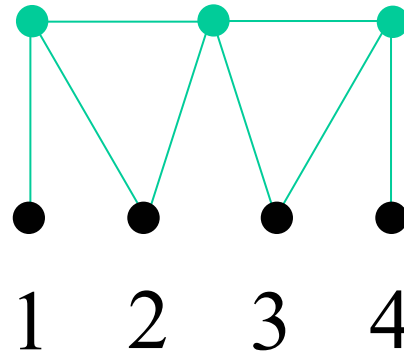
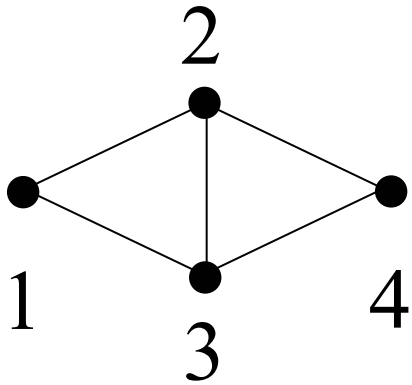
Simplicial Powers

A finite undirected graph $G = (V_G, E_G)$ is the *k-simplicial power* of a graph $H = (V_H, E_H)$ if $V_G \subseteq V_H$ is the set of all **simplicial vertices of H** and for all distinct $x, y \in V_G$

$$xy \in E \Leftrightarrow \text{dist}_H(x, y) \leq k.$$

H is called a *k-simplicial root of G* .

Simplicial Powers



Trees vs. Block Graphs

The *line graph* $L(G)$ of $G=(V,E)$ is the *edge intersection graph of G* , i.e., E is the set of nodes of $L(G)$, and distinct $e,e' \in E$ are adjacent in $L(G)$ if $e \cap e' \neq \emptyset$.

Theorem [Harary, 1972]

G is the line graph of a tree \Leftrightarrow

G is a claw-free block graph.

Trees vs. Block Graphs

Theorem [B., Le, 2008] For any $k \geq 2$,

G is the k -leaf power of a tree \Leftrightarrow

G is the $(k-1)$ -simplicial power of a claw-free block graph.

Trees vs. Block Graphs

Buneman's 4-point condition:

For every four distinct vertices u, v, x, y ,
the following holds:

$$(*) \operatorname{dist}_H(u, v) + \operatorname{dist}_H(x, y) \leq \max \{ \\ \operatorname{dist}_H(u, x) + \operatorname{dist}_H(v, y), \operatorname{dist}_H(u, y) + \operatorname{dist}_H(v, x) \}$$

Trees vs. Block Graphs

Theorem [Buneman 1974, Howorka 1979]

Let G be a connected graph.

- (i) G is a **tree** if and only if G is Δ -free and fulfills the 4-point condition (*).
- (ii) G is a **block graph** if and only if G fulfills the 4-point condition (*).

Every graph is a simplicial power

Theorem

Every graph is the 2-simplicial power of a split graph.

Every graph is a simplicial power

Theorem

Every graph is the 2-simplicial power of a split graph.

2-SIMPLICIAL SPLIT GRAPH ROOT:

Instance: A graph G and an integer k .

Question: Is there a split graph H with $\leq k$ nodes such that G is the 2-simplicial power of H ?

Every graph is a simplicial power

Theorem

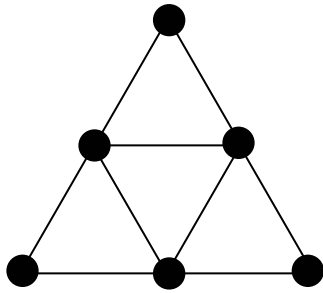
Every graph is the 2-simplicial power of a split graph.

2-SIMPLICIAL SPLIT GRAPH ROOT:

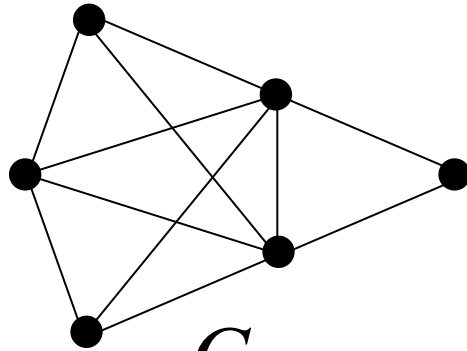
Instance: A graph G and an integer k .

Question: Is there a split graph H with $\leq k$ nodes such that G is the 2-simplicial power of H ?

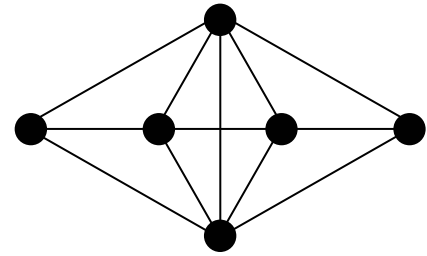
Theorem The problem 2-SIMPLICIAL SPLIT GRAPH ROOT is NP-complete.



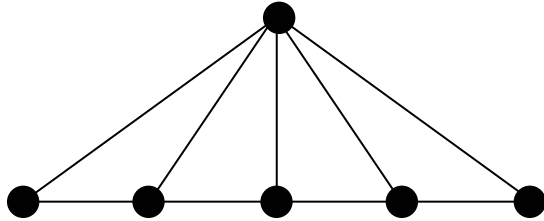
G_1



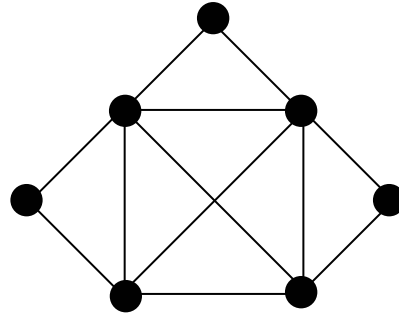
G_2



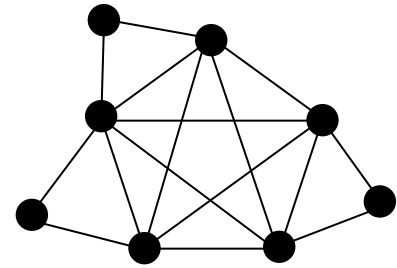
G_3



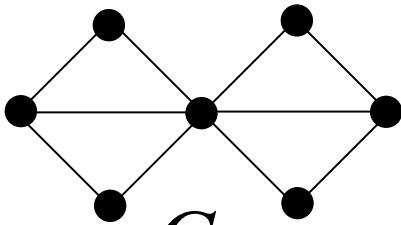
G_4



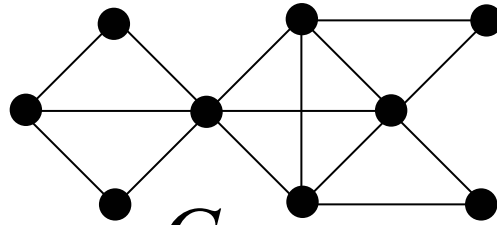
G_5



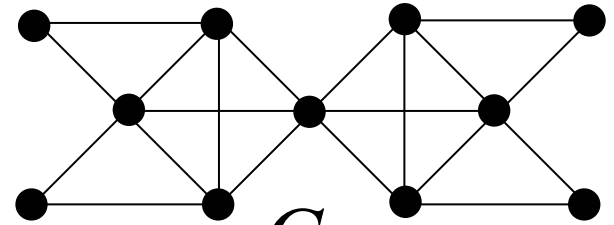
G_6



G_7



G_8



G_9

3-Simplicial Powers of Block Graphs

Theorem

The following conditions are equivalent:

- (i) G is the 3-simplicial power of a block graph.
- (ii) G is an induced subgraph of the square of a block graph.
- (iii) Each block of G is a basic 3-simplicial power of a block graph, and each cut vertex v of G is non-special in at most one block containing v .
- (iv) G is a basic (6,8)-leaf power.
- (v) G is (G_1, \dots, G_9) -free chordal.

Open Problems

Problem 1

Recognition complexity of leaf powers (of k -leaf powers for $k \geq 6$)

Problem 2

Complexity of determining the leaf rank of a given leaf power (even open for unit interval graphs)

Thank you for your attention!

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