Skolem sequences: from coloured blocks to communication networks

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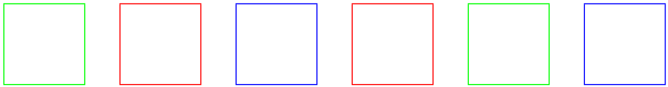
Coloured blocks

Years ago, my son, then a little boy, was playing with some coloured blocks. There were two of each colour, and one day I noticed that he had placed them in a single pile so that between the red pair there was one block, two between the blue pair, and three between the yellow...

C. Dudley Langford, Math. Gazette 1958



Skolem-type sequences
definitions
existence
Designs
Graphs

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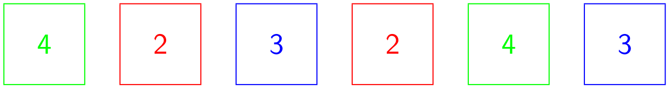

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an integer sequence with

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- in positions $\mathcal{P}$
- each entry $i \in \mathcal{E}$ appears exactly twice in positions $a_i, a_i + i \in \mathcal{P}$ (Skolem property)

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$\mathcal{E} = \{1, 2, 3, 4\}$, $\mathcal{P} = \{1, \ldots, 8\}$

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Langford	-	$\mathcal{E} = \{d, \ldots, n\}$, for some $d > 1$

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Existence - Skolem

Theorem (Baker 1995)

A k-extended Skolem sequence of order n exists iff

1. $n \equiv 0, 1 \pmod 4$ *and k is odd; or*
2. $n \equiv 2, 3 \pmod 4$ *and k is even.*

Existence - Rosa

Theorem (Linek and Shalaby 2008)

There exists a (p, q)*-Rosa sequence of order* n *if and only if* $(n, p, q) \neq (1, 2, 3), (4, 5, 6)$ *and either*

1. $n \equiv 0, 1 \pmod 4$ *and* p *and* q *have opposite parity, or*
2. $n \equiv 2, 3 \pmod 4$ *and* p *and* q *have the same parity.*

Existence - Langford

Theorem (Simpson 1983)

A Langford sequence of defect d and length m (order $n = d + m - 1$) exists if and only if $m \geq 2d - 1$ and either

1. $m \equiv 0, 1 \pmod 4$ *and d is odd, or*
2. $m \equiv 0, 3 \pmod 4$ *and d is even.*

Existence - near

Theorem

If there exists a k-extended q-near Skolem sequence of order n, then $(n, q, k) \neq (3, 2, 3), (4, 2, 4)$ *and either*

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These conditions are sufficient if $k = 2n - 1, 2n - 2$.

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Conjecture

These necessary conditions are sufficient.

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1 in positions 7 and 8 $\longrightarrow$ $(0, 1, 12)$

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For $n \equiv 2, 3 \pmod 4$, $hS_n \longrightarrow$ cyclic $STS(6n+1)$

An adaptation (Rosa)

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For $n \equiv 1, 2 \pmod 4$, use $(n+1, 2n+1) - \mathcal{R}_n$.

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PTS is maximal unless $k = n+1$.

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$(0, 1, 10)$, $(0, 2, 8)$, $(0, 3, 7)$ cycled mod 21

The leave consists of $\{(c, c+5) : c = 0, \ldots, 20\}$.

Even more cyclic STS

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$(0,1,10)$, $(0,2,6)$, $(0,3,11)$, $(0,5,12)$

This STS is new unless $n+1$ is in the first position of the sequence.

Even more cyclic STS

Can insert more consecutive zeros [McDonald 2002]:

$(2,\ldots,m+2)$-ext $(n-m)$-near S_{n+1}	$\longrightarrow$	$STS(6n+1)$
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a and $a+i$

4 0 0 5 4 1 1 2 5 2

$(0, 1, 9)$, $(0, 2, 12)$, $(0, 4, 7)$, $(0, 5, 11)$

BSEC($v, 3, \lambda$)

Balanced sampling plan excluding contiguous units

- vertex set has a cyclic ordering
- no 2 contiguous vertices are in a block
- 2 noncontiguous vertices are in exactly λ blocks

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$$\mathcal{L}_2^{n-1} \longrightarrow \text{cyclic } BSEC(6n-3, 3, 1) \quad \text{[Wei 2002]}$$

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$\mathcal{L}_2^{2t-1} \longrightarrow$ cyclic $BSEC(6t, 3, 2)$ [Wei 2002]

$\mathcal{L}_d^m \longrightarrow$ cyclic $BSEC(v, 3, \lambda)$ [Zhang and Chang 2005]

BSA$(v, 3, \lambda, \alpha)$

Balanced sampling plan excluding adjacent units

- avoid using pairs of points located within α units of one another

$$\mathcal{L}^{t}_{\alpha+1} \longrightarrow \begin{array}{l} \text{cyclic } BSA(6t + 2\alpha + 1, 3, 1, \alpha) \\ \text{all } t \geq 2\alpha + 1 \end{array}$$

[Wei 2002]

Graphs

Skolem-type sequences have been used to:

- construct other types of labelings (e.g., graceful, ordered graceful)
- directly label graphs

Skolem labeling

Design a scheme to check a communications network for

- node
- link
- distance

reliability. [Mendelsohn and Shalaby, 1991]

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Graph on $2n$ vertices is Skolem labeled if

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Labeling is strong if every edge is essential.

Which families of Skolem labeled graphs have been characterized?

- paths and cycles [Mendelsohn and Shalaby 1991]
- ladder graphs [Baker, Bonato and Kergin 2002]
- $P_s \square P_t$ [Graham, Pike and Shalaby 2007]
- 3-windmills [Mendelsohn and Shalaby 1999]
- generalized 3-windmills [Baker and Manzer 2008]

For trees

The Skolem parity of a vertex u of a tree $T = (V, E)$ is

$$\sum_{v \in V} d(u, v)(mod\ 2).$$

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A Skolem labeled tree T on $2n$ vertices satisfies

- Skolem parity condition: either
  1. the Skolem parity of T is even and $n \equiv 0, 3 \pmod 4$ or
  2. the Skolem parity of T is odd and $n \equiv 1, 2 \pmod 4$.
- Non-degeneracy condition

Conjecture

The necessary conditions are sufficient.

Thanks

- NSERC
- Mount Allison University
- all those who have worked on these sequences and their applications
- you, for your attention