On packing bases in matroids

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- We use this one:
  A **matroid** is a pair $M = (E, \mathcal{I})$, where E is a finite set and $\mathcal{I}$ is a family of subsets of E satisfying:
  (i) $\emptyset \in \mathcal{I}$;
  (ii) for all $I \in \mathcal{I}$, if $J \subseteq I$ then $J \in \mathcal{I}$;
  (iii) if $I, J \in \mathcal{I}$ with $|J| < |I|$, then $\exists x \in I \setminus J$ s.t. $J \cup \{x\} \in \mathcal{I}$.

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- A maximal element of $\mathcal{I}$ is called a **base** of the matroid M.
  The (constant) size of a base is called the **rank** of M.

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- For example, for the cycle matroid $M[G]$ of a connected graph G, this is the *spanning tree packing number* of G.
- Edmonds (1965) proved that if $\sigma(M) \geq k$, for all $X \subseteq E$, we have

$$k.\text{rank}(X) + |E \setminus X| \geq k.\text{rank}(M).$$

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- The **cogirth** of M, denoted $\lambda(M)$ is the smallest size of a cocircuit.
- The cogirth of the graphic matroid $M[G]$ is just the edge connectivity of G.

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- Given this bound, it is natural to ask: for which matroids are they equal?
- For (connected) graphs, this question was addressed by B. Stevens and the speaker (preprint, 01/2009). We provided a structural description of such graphs.

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- Note that this partitions the class of matroids with $\sigma(M) \geq k$ into two parts.

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- **Lemma:** Suppose $\sigma(M) = \lambda(M) = k$. For any two cocircuits C_1, C_2 of M, $C_1 \cap C_2 = \emptyset$.
- *Proof:* by contradiction, using an argument about rank.

The crux of the problem?

- For a matroid M with $\lambda(M) = \sigma(M) = k$, with k-cocircuits $C_1, \ldots, C_t$, the **crux** of M is defined to be

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  - $\text{crux}(M)$ has rank $r - t$ (where $r = \text{rank}(M)$);
  - $\text{crux}(M)$ also has k disjoint bases.
- If M is a connected matroid, $\text{crux}(M)$ may not be connected. (It may even be empty.) We denote the number of connected components of $\text{crux}(M)$ by $\delta(M)$.

Assembly matrices

- Suppose M is a connected matroid such that $\sigma(M) = \lambda(M) = k$, with t cocircuits $C_1, \ldots, C_t$, and where $\text{crux}(M)$ has d connected components $M_1, \ldots, M_d$.

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- Think of A as a sort of "incidence matrix".

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- Then we have:
    1. There exists a unique set of k-irreducible matroids $\mathcal{M} = \{M_1, \dots, M_m\}$ (for some integer m).
    2. There exists a unique rooted tree R with m leaves labelled by $M_1, \dots, M_m$, such that the root is labelled by M and each non-leaf labelled by K has $d = \delta(K)$ children, labelled by the connected components of crux(K).

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  2. There exists a unique rooted tree R with m leaves labelled by $M_1, \ldots, M_m$, such that the root is labelled by M and each non-leaf labelled by K has $d = \delta(K)$ children, labelled by the connected components of $\mathrm{crux}(K)$.
  3. For each non-leaf, labelled by K and its d children labelled $K_1, \ldots, K_d$, there exists a unique $t \times d$ assembly matrix, and where $\sum_{i=1}^{d} \mathrm{rank}(K_i) = \mathrm{rank}(K) - t$.

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- The rooted tree R and the collection of assembly matrices tell you how to do it.

THE END