

On packing bases in matroids

Robert Bailey
Carleton University, Ottawa

CanaDAM 2009
28th May 2009

Matroids

- ▶ There are various equivalent ways of defining a matroid.

Matroids

▶ There are various equivalent ways of defining a matroid.

▶ We use this one:

A **matroid** is a pair $M = (E, \mathcal{I})$, where E is a finite set and \mathcal{I} is a family of subsets of E satisfying:

- (i) $\emptyset \in \mathcal{I}$;
- (ii) for all $I \in \mathcal{I}$, if $J \subseteq I$ then $J \in \mathcal{I}$;
- (iii) if $I, J \in \mathcal{I}$ with $|J| < |I|$, then $\exists x \in I \setminus J$ s.t. $J \cup \{x\} \in \mathcal{I}$.

Matroids

- ▶ There are various equivalent ways of defining a matroid.
- ▶ We use this one:
A **matroid** is a pair $M = (E, \mathcal{I})$, where E is a finite set and \mathcal{I} is a family of subsets of E satisfying:
 - (i) $\emptyset \in \mathcal{I}$;
 - (ii) for all $I \in \mathcal{I}$, if $J \subseteq I$ then $J \in \mathcal{I}$;
 - (iii) if $I, J \in \mathcal{I}$ with $|J| < |I|$, then $\exists x \in I \setminus J$ s.t. $J \cup \{x\} \in \mathcal{I}$.
- ▶ A maximal element of \mathcal{I} is called a **base** of the matroid M .
The (constant) size of a base is called the **rank** of M .

Packing bases

- ▶ The **base packing number** of M , denoted $\sigma(M)$, is the size of the largest collection of mutually disjoint bases for M .

Packing bases

- ▶ The **base packing number** of M , denoted $\sigma(M)$, is the size of the largest collection of mutually disjoint bases for M .
- ▶ For example, for the cycle matroid $M[G]$ of a connected graph G , this is the *spanning tree packing number* of G .

Packing bases

- ▶ The **base packing number** of M , denoted $\sigma(M)$, is the size of the largest collection of mutually disjoint bases for M .
- ▶ For example, for the cycle matroid $M[G]$ of a connected graph G , this is the *spanning tree packing number* of G .
- ▶ Edmonds (1965) proved that if $\sigma(M) \geq k$, for all $X \subseteq E$, we have

$$k \cdot \text{rank}(X) + |E \setminus X| \geq k \cdot \text{rank}(M).$$

Cocircuits

- ▶ A **cocircuit** of M is a minimal subset of E whose deletion reduces the rank of M .

Cocircuits

- ▶ A **cocircuit** of M is a minimal subset of E whose deletion reduces the rank of M .
- ▶ Equivalently, a cocircuit is a minimal subset of E which intersects every base of M .

Cocircuits

- ▶ A **cocircuit** of M is a minimal subset of E whose deletion reduces the rank of M .
- ▶ Equivalently, a cocircuit is a minimal subset of E which intersects every base of M .
- ▶ Equivalently, a cocircuit is the complement of a hyperplane of M .

Cocircuits

- ▶ A **cocircuit** of M is a minimal subset of E whose deletion reduces the rank of M .
- ▶ Equivalently, a cocircuit is a minimal subset of E which intersects every base of M .
- ▶ Equivalently, a cocircuit is the complement of a hyperplane of M .
- ▶ The **cogirth** of M , denoted $\lambda(M)$ is the smallest size of a cocircuit.

Cocircuits

- ▶ A **cocircuit** of M is a minimal subset of E whose deletion reduces the rank of M .
- ▶ Equivalently, a cocircuit is a minimal subset of E which intersects every base of M .
- ▶ Equivalently, a cocircuit is the complement of a hyperplane of M .
- ▶ The **cogirth** of M , denoted $\lambda(M)$ is the smallest size of a cocircuit.
- ▶ The cogirth of the graphic matroid $M[G]$ is just the edge connectivity of G .

Base packing and cogirth

- ▶ For graphs, it is easy to see that the s.t.p. number is bounded above by the edge connectivity.

Base packing and cogirth

- ▶ For graphs, it is easy to see that the s.t.p. number is bounded above by the edge connectivity.
- ▶ This carries over to matroids: for any matroid M , we have

$$\sigma(M) \leq \lambda(M).$$

Base packing and cogirth

- ▶ For graphs, it is easy to see that the s.t.p. number is bounded above by the edge connectivity.
- ▶ This carries over to matroids: for any matroid M , we have

$$\sigma(M) \leq \lambda(M).$$

- ▶ Given this bound, it is natural to ask: for which matroids are they equal?

Base packing and cogirth

- ▶ For graphs, it is easy to see that the s.t.p. number is bounded above by the edge connectivity.
- ▶ This carries over to matroids: for any matroid M , we have

$$\sigma(M) \leq \lambda(M).$$

- ▶ Given this bound, it is natural to ask: for which matroids are they equal?
- ▶ For (connected) graphs, this question was addressed by B. Stevens and the speaker (preprint, 01/2009). We provided a structural description of such graphs.

Reducibility

- ▶ The following notion will help us state our theorem:

Reducibility

- ▶ The following notion will help us state our theorem:
- ▶ **Definition:** Suppose M is a matroid with $\sigma(M) \geq k$. Then:

Reducibility

- ▶ The following notion will help us state our theorem:
- ▶ **Definition:** Suppose M is a matroid with $\sigma(M) \geq k$. Then:
 - ▶ we call M **k -reducible** if $\lambda(M) = \sigma(M) = k$;

Reducibility

- ▶ The following notion will help us state our theorem:
- ▶ **Definition:** Suppose M is a matroid with $\sigma(M) \geq k$. Then:
 - ▶ we call M **k -reducible** if $\lambda(M) = \sigma(M) = k$;
 - ▶ we call M **k -irreducible** otherwise (i.e. if either $\sigma(M) > k$ or $k = \sigma(M) < \lambda(M)$).

Reducibility

- ▶ The following notion will help us state our theorem:
- ▶ **Definition:** Suppose M is a matroid with $\sigma(M) \geq k$. Then:
 - ▶ we call M **k -reducible** if $\lambda(M) = \sigma(M) = k$;
 - ▶ we call M **k -irreducible** otherwise (i.e. if either $\sigma(M) > k$ or $k = \sigma(M) < \lambda(M)$).
- ▶ Note that this partitions the class of matroids with $\sigma(M) \geq k$ into two parts.

No disjoint cocircuits

- ▶ An important lemma is this:

No disjoint cocircuits

- ▶ An important lemma is this:
- ▶ **Lemma:** Suppose $\sigma(M) = \lambda(M) = k$. For any two cocircuits C_1, C_2 of M , $C_1 \cap C_2 = \emptyset$.

No disjoint cocircuits

- ▶ An important lemma is this:
- ▶ **Lemma:** Suppose $\sigma(M) = \lambda(M) = k$. For any two cocircuits C_1, C_2 of M , $C_1 \cap C_2 = \emptyset$.
- ▶ *Proof:* by contradiction, using an argument about rank.

The crux of the problem?

- ▶ For a matroid M with $\lambda(M) = \sigma(M) = k$, with k -cocircuits C_1, \dots, C_t , the **crux** of M is defined to be

$$\text{crux}(M) = M \setminus \bigcup_{i=1}^t C_i.$$

The crux of the problem?

- ▶ For a matroid M with $\lambda(M) = \sigma(M) = k$, with k -cocircuits C_1, \dots, C_t , the **crux** of M is defined to be

$$\text{crux}(M) = M \setminus \bigcup_{i=1}^t C_i.$$

- ▶ Two observations about the crux:
 - ▶ $\text{crux}(M)$ has rank $r - t$ (where $r = \text{rank}(M)$);
 - ▶ $\text{crux}(M)$ also has k disjoint bases.

The crux of the problem?

- ▶ For a matroid M with $\lambda(M) = \sigma(M) = k$, with k -cocircuits C_1, \dots, C_t , the **crux** of M is defined to be

$$\text{crux}(M) = M \setminus \bigcup_{i=1}^t C_i.$$

- ▶ Two observations about the crux:
 - ▶ $\text{crux}(M)$ has rank $r - t$ (where $r = \text{rank}(M)$);
 - ▶ $\text{crux}(M)$ also has k disjoint bases.
- ▶ If M is a connected matroid, $\text{crux}(M)$ may not be connected. (It may even be empty.) We denote the number of connected components of $\text{crux}(M)$ by $\delta(M)$.

Assembly matrices

- ▶ Suppose M is a connected matroid such that $\sigma(M) = \lambda(M) = k$, with t cocircuits C_1, \dots, C_t , and where $\text{crux}(M)$ has d connected components M_1, \dots, M_d .

Assembly matrices

- ▶ Suppose M is a connected matroid such that $\sigma(M) = \lambda(M) = k$, with t cocircuits C_1, \dots, C_t , and where $\text{crux}(M)$ has d connected components M_1, \dots, M_d .
- ▶ The *assembly matrix* of M , is the $d \times t$ matrix A , with entries 0 or 1, such that:

Assembly matrices

- ▶ Suppose M is a connected matroid such that $\sigma(M) = \lambda(M) = k$, with t cocircuits C_1, \dots, C_t , and where $\text{crux}(M)$ has d connected components M_1, \dots, M_d .
- ▶ The *assembly matrix* of M , is the $d \times t$ matrix A , with entries 0 or 1, such that:
- ▶ for all i, j , the restriction of M to

$$C_j \cup \left(\bigcup_{i: A_{ij}=1} M_i \right)$$

is the unique maximal connected component of $C_j \cup \text{crux}(M)$.

Assembly matrices

- ▶ Suppose M is a connected matroid such that $\sigma(M) = \lambda(M) = k$, with t cocircuits C_1, \dots, C_t , and where $\text{crux}(M)$ has d connected components M_1, \dots, M_d .
- ▶ The *assembly matrix* of M , is the $d \times t$ matrix A , with entries 0 or 1, such that:
- ▶ for all i, j , the restriction of M to

$$C_j \cup \left(\bigcup_{i: A_{ij}=1} M_i \right)$$

is the unique maximal connected component of $C_j \cup \text{crux}(M)$.

- ▶ Think of A as a sort of “incidence matrix”.

Main Theorem

- ▶ Suppose M is a connected matroid for which $\sigma(M) = \lambda(M) = k$.

Main Theorem

- ▶ Suppose M is a connected matroid for which $\sigma(M) = \lambda(M) = k$.
- ▶ Then we have:

Main Theorem

- ▶ Suppose M is a connected matroid for which $\sigma(M) = \lambda(M) = k$.
- ▶ Then we have:
 1. There exists a unique set of k -irreducible matroids $\mathcal{M} = \{M_1, \dots, M_m\}$ (for some integer m).

Main Theorem

- ▶ Suppose M is a connected matroid for which $\sigma(M) = \lambda(M) = k$.
- ▶ Then we have:
 1. There exists a unique set of k -irreducible matroids $\mathcal{M} = \{M_1, \dots, M_m\}$ (for some integer m).
 2. There exists a unique rooted tree R with m leaves labelled by M_1, \dots, M_m , such that the root is labelled by M and each non-leaf labelled by K has $d = \delta(K)$ children, labelled by the connected components of $\text{crux}(K)$.

Main Theorem

- ▶ Suppose M is a connected matroid for which $\sigma(M) = \lambda(M) = k$.
- ▶ Then we have:
 1. There exists a unique set of k -irreducible matroids $\mathcal{M} = \{M_1, \dots, M_m\}$ (for some integer m).
 2. There exists a unique rooted tree R with m leaves labelled by M_1, \dots, M_m , such that the root is labelled by M and each non-leaf labelled by K has $d = \delta(K)$ children, labelled by the connected components of $\text{crux}(K)$.
 3. For each non-leaf, labelled by K and its d children labelled K_1, \dots, K_d , there exists a unique $t \times d$ assembly matrix, and where $\sum_{i=1}^d \text{rank}(K_i) = \text{rank}(K) - t$.

What the theorem really says

- ▶ Suppose $\lambda(M) = \sigma(M) = k$. Then M is “put together” from k -irreducible pieces in a controlled way.

What the theorem really says

- ▶ Suppose $\lambda(M) = \sigma(M) = k$. Then M is “put together” from k -irreducible pieces in a controlled way.
- ▶ The rooted tree R and the collection of assembly matrices tell you how to do it.

THE END