Algebraic Structure in a Family of Nim-like Arrays

Lowell Abrams
The George Washington University

Dena Cowen-Morton
Xavier University

CanaDaM 2009

Combinatorial Games

Basics

- full information
- no probability
- winning strategy
  (“1st player win”
  vs. “2nd player win”)

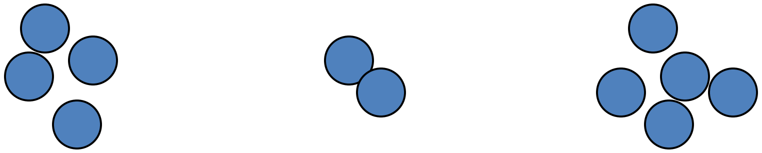
Combinatorial Games

Nim

even piles --- who wins?

Combinatorial Games

Nim



(this is a “sum” of three individual single-pile games)

Combinatorial Games

Nimbers and Nim addition

- Nim pile with n stones has nimber n
- nimber 0 means the second player wins
- two side-by-side Nim piles, both with nimber n, have sum 0:

  $$n+n = 0$$
- if $(r+s)+n = 0$ (*i.e.* is a second player win) then $r+s=n$

the Nimbers table
(G+H)

0	1	2	3	4	5	6	7
2	0	3	2	5	4	7	6
3	3	0	1	6	7	4	5
4	2	1	0	7	6	5	4
5	5	6	7	0	1	2	3
6	4	7	6	1	0	3	2
7	7	4	5	2	3	0	1
8	6	5	4	3	2	1	0

Rule: Seed with 0.

Rule: Enter smallest non-negative integer appearing neither above nor to left.

nother way to combine games

llman and Stromquist:
sequential compound G → H

isère play: G → 1

isère nim addition: G+H → 1

the Nimbers table
(G+H → 2)

2	0	1	3	4	5	6	7	8	9
0	1	2	4	3	6	5	8	7	10
1	2	0	5	6	3	4	9	10	7
3	4	5	0	1	2	7	6	9	8
4	3	6	1	0	7	2	5	11	12
5	6	3	2	7	0	1	4	12	11
6	5	4	7	2	1	0	3	13	14
7	8	9	6	5	4	3	0	1	2
8	7	10	9	11	12	13	1	0	3
9	10	7	8	12	11	14	2	3	0

Rule: Seed with 2.

Proceed with same algorithm.

An algebraic approach...

view array as defining an operation ¤ on $N_{,0}$

2	0	1	3	4	5	6
0	1	2	4	3	6	5
1	2	0	5	6	3	4
3	4	5	0	1	2	7
4	3	6	1	0	7	2
5	6	3	2	7	0	1

3 ¤ 3 = 0

4 ¤ 5 = 7

Basic algebraic structure

view array as defining an operation ¤ on $N_{,0}$

2	0	1	3	4	5	6
0	1	2	4	3	6	5
1	2	0	5	6	3	4
3	4	5	0	1	2	7
4	3	6	1	0	7	2
5	6	3	2	7	0	1

¤ is commutative

2 is the ¤-identity

¤ is *not* associative

e.g.

1 ¤ (1 ¤ 4) = 1 ¤ 6 = 4

(1 ¤ 1) ¤ 4 = 0 ¤ 4 = 3

have $A_2 = (N_{,0}, ¤)$

write: A_s , by analogy, for each seed s

Basic algebraic structure, continued…

“(Q,¤) is a **quasigroup**” means:

for every i,j2 Q
there exist unique p,q 2 Q
such that i¤p = j and q¤i = j

“(Q,¤) is a **loop**” means:

(Q,¤) is a quasigroup
with a two-sided ¤-identity

Quasigroups

all groups are quasigroups

x	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

(units in **Z**/5**Z**, under multiplication)

but
not every quasigroup
is a group

/	1	2	3	4
1	1	2	3	4
2	3	1	4	2
3	2	4	1	3
4	4	3	2	1

(units in **Z**/5**Z**, under division)

note: 2/(3/2) = 2/4 = 2 but (2/3)/2 = 4/2 = 3

Basic algebraic structure, continued...

“(Q,¤) is a **quasigroup”** means:

for every i,j2 Q
there exist unique p,q 2 Q
such that i¤p = j and q¤i = j

“(Q,¤) is a **loop”** means:

(Q,¤) is a quasigroup
with a two-sided ¤-identity

observe: A_s is a loop

Take-Home Point:

Algebraic results

provide a way
to encode

combinatorial properties

Main Results (in brief)

<u>Theorem</u>

For each seed $s \geq 2$, A_s is monogenic.

<u>Theorem</u>

There are no nontrivial homomorphisms $A_s \to A_t$
if $s \geq 2$ or $t \geq 2$.
Otherwise, there are a lot of them.

Monogenicity

<u>Notation</u>: $«x; \lozenge»$ is the free unital groupoid on generator x with operation $\lozenge$

Note, e.g. : $(x \lozenge x) \lozenge (x \lozenge x) \neq x \lozenge (x \lozenge (x \lozenge x))$

Write x^n for $\underbrace{x \lozenge (\ldots \lozenge (x \lozenge x))}_{n \text{ times}}$

Monogenicity

loop L , element $n \in L$

define $\varphi_n : \text{«}x; \Diamond\text{»} \to L$

- operation-preserving
- $\varphi_n(e_\Diamond) = e_L$
- $\varphi_n(x) = n$

define L is *monogenic*: there is $n \in L$
such that φ_n is surjective

note: this differs a little from the standard definition...

Monogenicity

Theorem (A. and Cowen-Morton)

A_s is monogenic iff $s \geq 2$

For $s=2$, every element $n>2$ is a generator.

For $s>2$, every element $n \neq s$ is a generator.

apparently, a novelty in the literature

Homomorphisms

Theorem (A. and Cowen-Morton)

The only loop homomorphism
$f{:}A_s \rightarrow A_t$
for $s \neq t$ and either $s \geq 2$ or $t \geq 2$ (or both)
is the trivial map $A_s \rightarrow \{t\}$.

For $s=t \geq 2$, homomorphism f is
either the trivial map $A_s \rightarrow \{s\}$
or the identity map.

Homomorphisms

Terri Evans (1953):

description of homomorphisms
of finitely presented monogenic loops

Theorem (A. and Cowen-Morton)
For any seed s,
the loop A_s is not finitely presented.

Homomorphisms

Essence of proof

- monogenicity
- commutativity of this diagram:

$$\begin{tikzcd}
 & \text{«}x; \Diamond\text{»} \arrow[dl, "\varphi_n"'] \arrow[dr, "\psi_{f(n)}"] & \\
A_s \arrow[rr, "f"'] & & A_t
\end{tikzcd}$$

(ψ is the appropriate evaluation map)

Homomorphisms

case: $s = t = 2$ and $f(3) > 6$

set

$$\alpha = (x^2)^2 \lozenge \left[x^2 \lozenge ((x^2)^2 \lozenge x)\right]$$

$$\beta = \left((x^2)^2 \lozenge x\right) \lozenge \left(x \lozenge [x^2 \lozenge ((x^2)^2 \lozenge x)]\right)$$

$$\begin{tikzcd}
 & \text{«}x;\ \lozenge\text{»} \arrow[dl, "\varphi_3"'] \arrow[dr, "\varphi_{f(3)}"] & \\
A_2 \arrow[rr, "f"'] & & A_2
\end{tikzcd}$$

$$\begin{tikzcd}
 & \alpha \arrow[dl] \arrow[d] & \beta \arrow[dll] \arrow[dr] & \\
4 \arrow[r, "?"] & \varphi_{f(3)}(\alpha) & \neq & \varphi_{f(3)}(\beta)
\end{tikzcd}$$

Homomorphisms

case: $s = t = 2$ and $f(3) = 4, 5, \text{ or } 6$

set

$$\alpha = (x^2 \Diamond x) \Diamond \Big[(x^2)^2 \Diamond \big[x^2 \Diamond \Big(x \Diamond [(x^2)^2 \Diamond (x^2 \Diamond x)]\Big)\big]\Big]$$

$$\beta = \big[(x^2)^2 \Diamond (x^2 \Diamond x)\big] \Diamond \big[x^2 \Diamond \Big(x \Diamond [(x^2)^2 \Diamond (x^2 \Diamond x)]\Big)\big]$$

$$\begin{tikzcd} & \langle\!\langle x; \Diamond \rangle\!\rangle \arrow[dl, "\varphi_3"'] \arrow[dr, "\varphi_{f(3)}"] & \\ A_2 \arrow[rr, "f"'] & & A_2 \end{tikzcd}$$

$$\begin{tikzcd} \alpha \arrow[d] \arrow[dr] & \beta \arrow[dl] \arrow[drr] & & \\ 13 \arrow[r, "?"] & \varphi_{f(3)}(\alpha) & \neq & \varphi_{f(3)}(\beta) \end{tikzcd}$$

Homomorphisms

<u>case</u>: $s = 2, t = 0$

for $\delta \in$ «x; ◊»
define $|\delta|$ = number of x’s in δ

$$\begin{tikzcd}
& \text{«}x;\, \Diamond\text{»} \arrow[dl, "\varphi_n"'] \arrow[dr, "\psi_{f(n)}"] & \\
A_2 \arrow[rr, "f"'] & & A_0
\end{tikzcd}$$

A_0 is asociative

for $\delta \in$ «x; ◊»,

$$f \circ \varphi_n(\delta) = \psi_{f(n)}(x^{|\delta|})$$

in A_0, $m^2=0$
for all m

$$= \begin{cases} f(n) \text{ if } |\delta| \equiv 1 \pmod 2 \\ 0 \quad \text{otherwise} \end{cases}$$

Homomorphisms

<u>case</u>: $s = 2,\ t = 0$

$$\begin{tikzcd}
 & \ll x;\ \Diamond \gg \arrow[dl, "\varphi_3"'] \arrow[dr, "\Psi_{f(3)}"] & \\
A_2 \arrow[rr, "f"'] & & A_0
\end{tikzcd}$$

set

$$\alpha = (x^2)^2 \Diamond \left[x \Diamond (x^3 \Diamond (x^2)^2)\right]$$

$$\beta = x \Diamond \left(x^2 \Diamond \left[x \Diamond (x^3 \Diamond (x^2)^2)\right] \right)$$

since 3 generates A_2 and 0 is the identity in A_{0}, f is trivial

then we have

$$0 = f \circ \varphi_3(\alpha) = f \circ \varphi_3(\beta) = f(3)$$

$|\alpha| = 12$

$\varphi_3(\alpha) = 9 = \varphi_3(\beta)$

$|\beta| = 11$

Homomorphisms

<u>Theorem</u> (A. and Cowen-Morton)

$$\mathrm{Hom}(A_0,A_0) = \prod_{\geq 0} A_0$$

$$\mathrm{Hom}(A_0,A_1) = \prod_{\geq 0} \mathbf{Z}/2\mathbf{Z}$$

$$\mathrm{Hom}(A_1,A_0) = \prod_{\geq 0} A_0$$

$$\mathrm{Hom}(A_1,A_1) = \prod_{\geq 1} \mathbf{Z}/2\mathbf{Z} \ [\ \Big[\mathrm{Inj}(A_0,A_0) \ \pounds \ \{0,1\}^{\mathrm{N}}\Big]$$

Homomorphisms

behind the proof...

Each element 2^i in A_0 $(i \geq 0)$ generates a subgroup H_i isomorphic to $\mathbf{Z}/2\mathbf{Z}$.

A_0 is the weak product of the H_i since its operation is bitwise XOR.

Each element 2^i in A_1 $(i \geq 1)$ generates a subgroup $G_i = \{2^i, 0, 2^i+1, 1\}$ isomorphic to $\mathbf{Z}/4\mathbf{Z}$.

A_1 is *not* the weak product of the G_i but the G_i stay out of each other's way.

Homomorphisms

behind the
proof...

Theorem (A. and Cowen-Morton)

Let Q1 denote the loop quotient of A_1 by the relation $0 \equiv 1$.

Let Q2 denote the loop quotient of A_1 by the relations $\{2k \equiv 2k+1 \mid k = 1, 2, \ldots\}$.

Let Q3 denote the loop quotient of A_1 by all relations enforcing associativity.

Then each of these quotients is isomorphic to A_0 under an isom'm sending G_i to H_{i-1} for each i, for which all three quotient maps are the same.

Homomorphisms

<u>Theorem</u> (A. and Cowen-Morton)

$$\mathrm{Hom}(A_0,A_0) = \prod_{\geq 0} A_0$$

$$\mathrm{Hom}(A_0,A_1) = \prod_{\geq 0} \mathbf{Z}/2\mathbf{Z}$$

$$\mathrm{Hom}(A_1,A_0) = \prod_{\geq 0} A_0$$

$$\mathrm{Hom}(A_1,A_1) = \prod_{\geq 1} \mathbf{Z}/2\mathbf{Z} \ [\ \left[\mathrm{Inj}(A_0,A_0) \pounds \{0,1\}^{\mathrm{N}}\right]$$

◀