Additive Combinatorics
March 30 – April 12, 2006

On a Kakeya-type problem

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Abstract

Let A be a finite set of integers $A = \{a_1, a_2, \ldots, a_k\}$, $k = |A|$. For every element b_i of the sumset $2A = A + A = \{a + a' \colon a \in A, a' \in A\} = \{b_1, b_2, \ldots, b_{|2A|}\}$ we denote $r_i = r_i(A) = |\{(a, a') \colon a + a' = b_i, a \in A, a' \in A\}|$ and

$$D_i = D_i(A) = \{a - a' \colon a \in A, a' \in A, a + a' = b_i\}, d_i = d_i(A) = |D_i|.$$

After an eventual reordering of the set $2A$, we may assume that $r_1 \geq r_2 \geq \cdots \geq r_{|2A|}$.. For every $1 \leq s \leq |2A|$ we denote $R_s(A) = |D_1 \cup D_2 \cup \cdots \cup D_s|$ and define

$$R_s(k) = \max\{R_s(A) \colon A \subseteq \mathbb{Z}, |A| = k\}.$$

Katz and Tao [Math. Res. Lett. **6** (1999), 625–630] obtained an estimate of $R_s(k)$ assuming $s \leq k$.

In this note we find the *exact value* of $R_s(k)$ in cases $s = 1$, $s = 2$ and $s = 3$. The case $s = 3$ appeared to be not so simple. The structure of extremal sets A^*, i.e. sets of integers for which we have $R_3(A^*) = R_3(k)$, led us to sets isomorphic to planar sets having a rather unexpected form of a perfect hexagon. The proof of this result suggests the way of dealing with the general case $s \geq 4$.