Additive Combinatorics
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Partion regularity of linear equations

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Abstract

In this talk, we will discuss several results and conjectures on partition regularity of linear equations. A linear homogeneous equation E is called *r-regular* if, for every r-coloring of the positive integers, there is a monochromatic solution to E. A linear homogeneous equation is called *regular* if it is r-regular for all positive integers r. In 1933, Rado conjectured that for each positive integer n, there is another positive integer $k(n)$ such that every linear homogeneous equation in n variables that is $k(n)$-regular is regular. Kleitman and I recently settled this conjecture in the case $n = 3$ by proving that every linear homogeneous equation in 3 variables that is 24-regular is regular.

A coloring of the nonzero rational numbers is minimal for an equation E if there are no monochromatic solutions to E and the coloring uses as few colors as possible. Let $E(n)$ denote the equation $x_0 + 2x_1 + \cdots + 2^{n-2}x_{n-2} = 2^{n-1}x_{n-1}$. Alexeev, Graham, and I conjecture that for $n > 2$, there is a unique n-coloring that is minimal for $E(n)$, and we have verified this conjecture for $n \in \{3, 4, 5, 6\}$. Our conjecture would imply that for $n > 1$, the equation $E(n + 1)$ is n-regular but not $(n + 1)$-regular, which would imply another conjecture of Rado.

Radoičić and I recently proved that for each number $q \in \mathbb{Q} \setminus \{-1, 0, 1\}$, the following statement is independent of Zermelo–Fraenkel set theory: every 3-coloring of the nonzero real numbers has a monochromatic solution to $x_0 + qx_1 = q^2x_2$.

A system L of linear homogeneous equations is called *countably regular* if, for every countable coloring of the real numbers, there is a monochromatic solution to L in distinct variables. I will present the classification in ZFC of countably regular systems of linear homogeneous equations. As a corollary, for each positive integer s, the equation $x_1 + sx_2 = x_3 + \cdots + x_{s+3}$ is countably regular if and only if $2^{\aleph_0} > \aleph_s$. This generalizes the case $s = 1$ due to Erdős and Kakutani.