Additive Combinatorics
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On the structure of sets with many "medium–size" arithmetic progressions

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Abstract

For $\mathcal{H}$ a finite subset of $\mathbb{R}$ and any $3 \le k \le |\mathcal{H}|$, we write

$$\mathcal{AP}(\mathcal{H}, k) := \#\{k\text{-term arithmetic progressions in } \mathcal{H}\}$$

and study the maximum possible number of k-term AP's that a set of n well-chosen elements can contain, i.e. the quantity

$$\mathcal{AP}(n, k) := \max_{|\mathcal{H}|=n} \mathcal{AP}(\mathcal{H}, k).$$

The Szemerédi–Trotter theorem implies the upper bound

$$\mathcal{AP}(n, k) \le C \cdot \frac{n^2}{k},$$

for an absolute constant $C > 0$. Also, this order of magnitude can really be attained, as shown by, say, arithmetic progressions of n terms as $\mathcal{H}$. Our goal is to answer the following natural question:

What is the structure of $\mathcal{H}$ *if* $\mathcal{AP}(\mathcal{H}, k)$ *attains the* (*optimal*) *order of magnitude* $|\mathcal{H}|^2/k$*?*

For small, fixed values of k the problem was first studied by Balog and Szemerédi who found that $\mathcal{H}$ must contain a good proportion of a generalized arithmetic progression. (Some related results on the number of copies similar to a given pattern were found by Erdős–Elekes, Laczkovich–Ruzsa and Abrègo–Elekes–Fernandez.) On the other hand, for large values, say $k \ge cn$ (for a fixed positive constant c and large n), $\mathcal{H}$ will by assumption contain a large arithmetic progression.

We completely characterize the structure of $\mathcal{H}$ in case of "intermediate" values of k — which may not exactly coincide with what one may expect.