Actions of Lie groups and
Lie algebras on manifolds

Morris W. Hirsch
Department of Mathematics
University of California, Berkeley
970 Evans Hall #3840
Berkeley, CA 94720-3840
USA

mwhirsch@chorus.net; hirsch@math.berkeley.edu

Abstract

This talk continues on a theme of Lie and later Mostow: Which Lie groups act effectively on which manifolds?

A $C^r$ action $\alpha$ of a finite dimensional real Lie algebra $\mathfrak{g}$ on an $n$-dimensional manifold $M$ means a linear map $\alpha$ from $\mathfrak{g}$ to the space of $C^r$ vector fields on $M$ which commutes with brackets if $r \geq 1$. When $r = 0$ it means a local action on $M$ of some Lie group having Lie algebra $\mathfrak{g}$. When $\dim M = n$ we write $M = M^n$ and call $\alpha$ an $n$-action.

The Epstein–Thurston Theorem says that if $\mathfrak{g}$ is solvable (respectively, nilpotent) and has an effective $n$-action then $\mathfrak{g}$ has derived length $\ell(\mathfrak{g}) \leq n + 1$ (resp., $\leq n$). This condition turns out to be far from sufficient for the existence of effective $n$-actions.

**Theorem:** For every $n \geq 2$ there is a nilpotent $\mathfrak{g}$ with $\ell(\mathfrak{g}) = 2$ having an effective affine action on $\mathbb{R}^{n+1}$. But every $n$-action of $\mathfrak{g}$ annihilates the center.

$\mathfrak{g}$ is nonsupersoluble if some element has a nonreal eigenvalue under the adjoint representation. All nonsolvable Lie algebras (and many solvable ones) have this property.

**Generalized Turiel Theorem:** Let $\beta$ be an effective of the direct product of $k \geq 1$ nonsupersoluble algebras on a compact manifold $N^{2k}$. If $\beta$ is real analytic then the Euler characteristic of $N^{2k}$ satisfies $\chi(N^{2k}) \geq \#\text{Fix}(\beta) \geq 0$. But this is false for $C^\infty$ actions.

**Spectral Rigidity Theorem:** Assume $X \in \mathfrak{g}$ and $\text{ad } X$ has $n$ eigenvalues that are linearly independent over the rationals. Let $\alpha$ be an effective analytic action of $\mathfrak{g}$ on $M^n$ and suppose $X^\alpha$ vanishes at $p \in M^n$. Given a neighborhood $U$ of $p$, there is a neighborhood $N$ of $\alpha$ in the space
of $C^1$ actions of $\mathfrak{g}$ on $M^n$, with the following property: If $\beta \in \mathcal{N}$ then $X^\beta$ vanishes at a unique $q \in U$, and $dX^\beta(q)$ has the same spectrum as $dX^\alpha(p)$. 