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Actions of Lie groups and Lie algebras on manifolds

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Abstract

This talk continues on a theme of Lie and later Mostow: Which Lie groups act effectively on which manifolds?

A C^r *action* α of a finite dimensional real Lie algebra $\mathfrak{g}$ on an n-dimensional manifold M means a linear map α from $\mathfrak{g}$ to the space of C^r vector fields on M which commutes with brackets if $r \geq 1$. When $r = 0$ it means a local action on M of some Lie group having Lie algebra $\mathfrak{g}$. When $\dim M = n$ we write $M = M^n$ and call α an n-action.

The **Epstein–Thurston Theorem** says that if $\mathfrak{g}$ is solvable (respectively, nilpotent) and has an effective n-action then $\mathfrak{g}$ has derived length $\ell(g) \leq n+1$ (resp., $\leq n$). This condition turns out to be far from sufficient for the existence of effective n-actions.

Theorem: *For every $n \geq 2$ there is a nilpotent $\mathfrak{g}$ with $\ell(\mathfrak{g}) = 2$ having an effective affine action on $\mathbb{R}^{n+1}$. But every n-action of $\mathfrak{g}$ annihilates the center.*

$\mathfrak{g}$ is *nonsupersoluble* if some element has a nonreal eigenvalue under the adjoint representation. All nonsolvable Lie algebras (and many solvable ones) have this property.

Generalized Turiel Theorem: Let β be an effective of the direct product of $k \geq 1$ nonsupersoluble algebras on a compact manifold N^{2k}. If β is real analytic then the Euler characteristic of N^{2k} satisfies $\chi(N^{2k}) \geq \#\mathrm{Fix}\,(\beta) \geq 0$. But this is false for C^∞ actions.

Spectral Rigidity Theorem: Assume $X \in \mathfrak{g}$ and ad X has n eigenvalues that are linearly independent over the rationals. Let α be an effective analytic action of $\mathfrak{g}$ on M^n and suppose X^α vanishes at $p \in M^n$. Given a neighborhood U of p, there is a neighborhood $\mathcal{N}$ of α in the space

of C^1 actions of $\mathfrak{g}$ on M^n, with the following property: If $\beta \in \mathcal{N}$ then X^β vanishes at a unique $q \in U$, and $dX^\beta(q)$ has the same spectrum as $dX^\alpha(p)$.