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Zeta functions of buildings and algebraic geometry

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This is a survey about the connections between combinatorial zeta functions defined for (hyper)graphs and zeta functions for algebraic varieties defined over finite fields. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Q}_p)$ be a discrete cocompact subgroup. Ihara introduced a zeta function $\zeta_\Gamma(s)$ in analogy with the Selberg zeta for a discrete cocompact $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$. Ihara's zeta has an Euler product, and remarkably is a rational function. Later this zeta was constructed in a purely combinatorial way in terms of the finite graph $X_\Gamma = \Gamma\backslash X$ where X is the Bruhat-Tits building (tree) for $\mathrm{SL}_2(\mathbb{Q}_p)$. All this was later generalized to define the zeta (and L) functions of any graph. Ihara's deepest discovery in this area is that often $\zeta_\Gamma(s)$ is essentially equal to the zeta function $Z(Y/\mathbb{F}_q, s)$ for a Shimura curve Y. If Y is any Shimura variety, we raise the general question as to whether $Z(Y/\mathbb{F}_q, s)$ may be similarly expressed in terms of combinatorial zeta functions of complexes such as $\Gamma\backslash X$ where X is the Bruhat-Tits building for suitable reductive groups G over local fields. There has been substantial recent progress in defining these zeta functions for $\mathrm{SL}_3(\mathbb{Q}_p)$ and $\mathrm{Sp}_4(\mathbb{Q}_p)$, by Winnie Li and her students, and we raise the problem to relate these to zeta functions of algebraic varieties.