ORBIT ALGEBRAS OF REPETITIVE CATEGORIES

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K	a field
A	finite dimensional K-algebra
$\operatorname{mod} A$	category of finite dimensional right A-modules
A^{op}	opposite algebra of A
$\operatorname{mod} A^{\text{op}}$	category of finite dimensional left A-modules

$D = \operatorname{Hom}_K(-, K)$

$$\operatorname{mod} A \underset{D}{\overset{D}{\rightleftarrows}} \operatorname{mod} A^{\text{op}} \quad \text{standard duality}$$

$\Gamma(\operatorname{mod} A)$ Auslander-Reiten quiver of A

$\tau_A = D\operatorname{Tr}$ Auslander-Reiten translation

A is **selfinjective** if A_A and ${}_AA$ are injective, equivalently the projective A-modules are injective

A is **Frobenius** if there exists a nondegenerate K-bilinear form $(-,-) : A \times A \to K$ such that $(a, bc) = (ab, c)$ for all $a, b, c \in K$

A is **symmetric** if $(-,-)$ is symmetric

A Frobenius $\Rightarrow$ A selfinjective

A basic, selfinjective $\Rightarrow$ A is Frobenius

A selfinjective $\iff$ A is Morita equivalent to a Frobenius algebra

A selfinjective $\xrightarrow{\text{Nakayama}}$ $\mathrm{soc}({}_AA) = \mathrm{soc}(A_A)$

Hence $\mathrm{soc}(A) := \mathrm{soc}({}_AA) = \mathrm{soc}(A_A)$ is an ideal of A

Selfinjective algebras A and Λ are **socle equivalent** if $A/\mathrm{soc}(A) \cong \Lambda/\mathrm{soc}(\Lambda)$

For selfinjective algebras:

$$\begin{array}{c}\textbf{Morita}\\\textbf{equivalence}\end{array} \Rightarrow \begin{array}{c}\textbf{derived}\\\textbf{equivalence}\end{array} \xRightarrow{\text{Rickard}} \begin{array}{c}\textbf{stable}\\\textbf{equivalence}\end{array}$$

Selfinjective algebras A and Λ are **derived equivalent** if the derived categories $D^b(\text{mod}\, A)$ and $D^b(\text{mod}\, \Lambda)$ are equivalent as triangulated categories.

Selfinjective algebras A and Λ are **stably equivalent** if the stable module categories (modulo projectives) $\underline{\text{mod}}A$ and $\underline{\text{mod}}\Lambda$ are equivalent.

A selfinjective

P indecomposable projective A-module

Then there is an Auslander-Reiten sequence

$0 \to \operatorname{rad} P \to (\operatorname{rad} P/\operatorname{soc} P) \oplus P \to P/\operatorname{soc} P \to 0$

in $\operatorname{mod} A$.

$\Gamma^s(\operatorname{mod} A)$ the **stable Auslander-Reiten quiver** of A (deleting the projective vertices in $\Gamma^s(\operatorname{mod} A)$ and the arrows attached to them)

We may recover $\Gamma(\operatorname{mod} A)$ from $\Gamma^s(\operatorname{mod} A)$ if we know the positions of $P/\operatorname{soc} P$, P indecomposable projective modules, in $\Gamma^s(\operatorname{mod} A)$.

Similarly, we may recover $\Gamma(\operatorname{mod} A)$ from $\Gamma(\operatorname{mod} A/\operatorname{soc} A)$

B basic, connected, finite dimensional K-algebra

$$\mathcal{E} = \left\{ e_i \;\middle|\; 1 \leq i \leq n \right\}$$

set of orthogonal primitive idempotents of B

$$1_B = e_1 + \cdots + e_n$$

$\widehat{B}$ **repetitive category** of B

$$\widehat{\mathcal{E}} = \left\{ e_{m,i} \;\middle|\; m \in \mathbb{Z}, 1 \leq i \leq n \right\} \text{ objects of } \widehat{B}$$

$$\widehat{B}(e_{m,i}, e_{r,j}) = \begin{cases} e_j B e_i & , r = m \\ D(e_i B e_j) & , r = m+1 \\ 0 & , \text{otherwise} \end{cases}$$

$\widehat{B}$ selfinjective locally bounded K-category

$\operatorname{mod} \widehat{B}$ category of finite dimensional right $\widehat{B}$-modules

$\operatorname{gl.dim} B < \infty \xrightarrow{\textbf{Happel}} \underline{\operatorname{mod}} \widehat{B} \cong D^b(\operatorname{mod} B)$ as triangulated K-categories

G group of K-linear automorphisms of $\widehat{B}$

G is **admissible** if G acts freely on $\widehat{\mathcal{E}}$ and has finitely many orbits

$\widehat{B}/G$ **orbit category** (**Gabriel**)

$\widehat{\mathcal{E}}/G$ set of objects of $\widehat{B}/G$

$$(\widehat{B}/G)(a,b) = \left\{ (f_{yx}) \in \prod_{x \in a, y \in b} \widehat{B}(x,y) \,\middle|\, g \cdot f_{yx} = f_{g(y),g(x)} \ \forall_{g \in G, x \in a, y \in b} \right\}$$

for all $a, b \in \widehat{\mathcal{E}}/G$.

$F : \widehat{B} \to \widehat{B}/G$ canonical **Galois covering**

$\widehat{\mathcal{E}} \ni x \mapsto Fx = G \cdot x \in \widehat{\mathcal{E}}/G$

$\forall_{x \in \widehat{\mathcal{E}}, a \in \widehat{\mathcal{E}}/G}$ F induces isomorphisms

$$\bigoplus_{Fy=a} \widehat{B}(x,y) \xrightarrow{\sim} (\widehat{B}/G)(Fx,a),$$

$$\bigoplus_{Fy=a} \widehat{B}(y,x) \xrightarrow{\sim} (\widehat{B}/G)(a,Fx).$$

$$\bigoplus(\widehat{B}/G) = \bigoplus_{a,b\in\widehat{\mathcal{E}}/G} (\widehat{B}/G)(a,b)$$

basic, connected, finite dimensional selfinjective K-algebra

We identify $\widehat{B}/G = \bigoplus \widehat{B}/G$

$\nu_{\widehat{B}} : \widehat{B} \to \widehat{B}$ **Nakayama automorphism** of $\widehat{B}$

$\nu_{\widehat{B}}(e_m, i) = e_{m+1,i}$ for all $(m,i) \in \mathbb{Z}\times\{1,\ldots,n\}$.

Then $G = (\nu_{\widehat{B}})$ is admissible and

$$\widehat{B}/(\nu_{\widehat{B}}) \cong \mathsf{T}(B) = B \ltimes D(B)$$

trivial extension of B by the B-B-bimodule $D(B)$

$\mathsf{T}(B) = B \oplus D(B)$ as K-vector space

$$(b,f)(b',f') = (bb', bf' + fb')$$

for $b, b' \in B$, $f, f' \in D(B)$

$\mathsf{T}(B)$ is a symmetric algebra,

$$\dim_K \mathsf{T}(B) = 2\dim_K B$$

More generally, for a positive integer r, we have

$$\mathsf{T}(B)^{(r)} = \widehat{B}/(\nu_{\widehat{B}}^{r}) =$$

$$= \left\{ \begin{bmatrix} b_1 & 0 & 0 & & & & \\ f_2 & b_2 & 0 & & & & \\ 0 & f_3 & b_3 & & & & \\ & & \ddots & \ddots & & & \\ & 0 & & f_{r-1} & b_{r-1} & 0 \\ & & & 0 & f_1 & b_1 \end{bmatrix} \\ b_1, \ldots, b_{r-1} \in B,\ f_1, \ldots, f_{r-1} \in D(B) \right\}$$

r-**fold trivial extension algebra** of B.

Then $\mathsf{T}(B) = \mathsf{T}(B)^{(1)} = (\mathsf{T}(B)^{(1)})^{\mathbb{Z}/r\mathbb{Z}}$ algebra of invariants.

An automorphism φ of $\widehat{B}$ is said to be

- **positive**, if for each pair $(m, i) \in \mathbb{Z} \times \{1, \ldots, n\}$, we have $\varphi(e_{m,i}) = e_{p,j}$ for some $p \geq m$ and some $j \in \{1, \ldots, n\}$;
- **rigid**, if for each pair $(m, i) \in \mathbb{Z} \times \{1, \ldots, n\}$, there exists $j \in \{1, \ldots, n\}$ such that $\varphi(e_{m,i}) = e_{m,j}$;
- **strictly positive** if it is positive but not rigid.

Theorem (Ohnuki–Takeda–Yamagata). *Let B be a basic, connected, finite dimensional K-algebra, φ a positive automorphism of $\widehat{B}$ and $A = \widehat{B}/(\varphi\nu_{\widehat{B}})$. Then A is symmetric if and only if $A \cong \mathsf{T}(B)$.*

There are many symmetric orbit algebras $\widehat{B}/(\varphi)$, where φ is a (strictly positive) root of $\nu_{\widehat{B}}$.

For example,

$$K[x]/(x^{n+1}) \cong \widehat{B}_n/(\varphi_n), \quad n \geq 1,$$

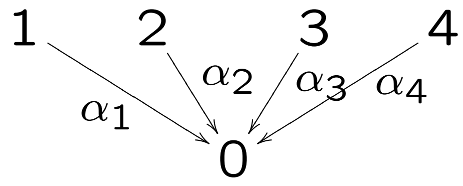
where $B_n = KQ_n$ the path algebra of the quiver

$$Q_n : 1 \longleftarrow 2 \longleftarrow \ldots \longleftarrow n$$

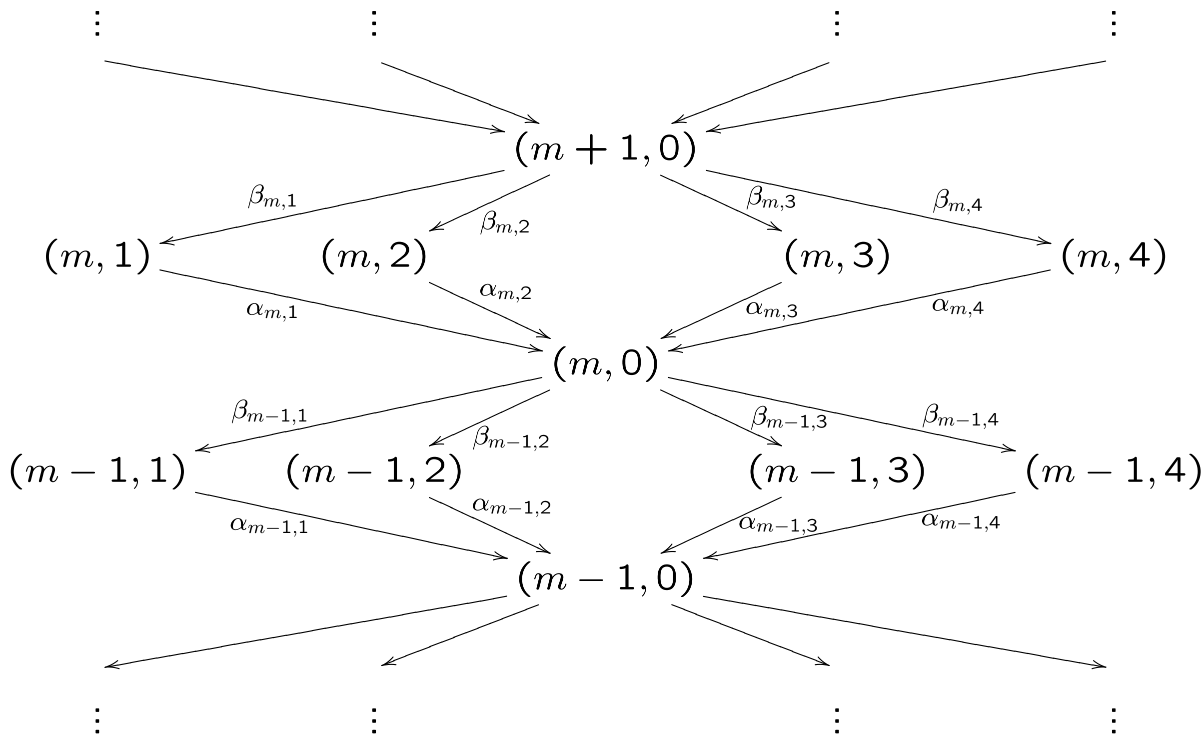
and φ_n strictly positive automorphism of $\widehat{B}$ with $\varphi_n^n = \nu_{\widehat{B_n}}$.

More generally, every **Brauer tree algebra** $A(T_S^n)$, T_S^n Brauer tree, S exceptional vertex of T_S^n with multiplicity n, is of the form $\widehat{B}/(\varphi)$ for a tilted algebra B of Dynkin type $\mathbb{A}_m$ (for some $m \geq 1$) and φ a strictly positive automorphism of $\widehat{B}$ with $\varphi^n = \nu_{\widehat{B}}$.

Example. Let Δ be the quiver



of Euclidean type $\widetilde{\mathbb{D}}_4$ and $B = K\Delta$ the path algebra of Δ over K. Then $\widehat{B} = K\widehat{\Delta}/\widehat{I}$ the bound quiver K-category, where $\widehat{\Delta}$ is of the form



and $\widehat{I}$ is the ideal of the path category $K\widehat{\Delta}$ of $\widehat{\Delta}$ generated by the elements

$$\beta_{m,i}\alpha_{m,i} - \beta_{m,j}\alpha_{m,j},\ \alpha_{m,i}\beta_{m-1,j},\ m \in \mathbb{Z},$$
$$i, j \in \{1,2,3,4\},\ i \neq j.$$

$e_{m,i}$ object of $\widehat{B}$ corresponding to $(m,i) \in \mathbb{Z} \times \{0,1,2,3,4\}$

$\nu_{\widehat{B}}(e_{m,i}) = e_{m+1,i}$ for $(m,i) \in \mathbb{Z} \times \{0,1,2,3,4\}$.

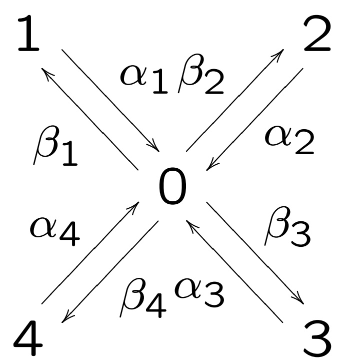
Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ be rigid automorphisms of $\widehat{B}$ given by

$$\sigma_1 = \left(e_{m,1}, e_{m,2}\right), \text{ for all } m \in \mathbb{Z},$$
$$\sigma_2 = \left(e_{m,1}, e_{m,2}, e_{m,3}\right), \text{ for all } m \in \mathbb{Z},$$
$$\sigma_3 = \left(e_{m,1}, e_{m,2}, e_{m,3}, e_{m,4}\right), \text{ for all } m \in \mathbb{Z},$$
$$\sigma_4 = \left(e_{m,1}, e_{m,2}\right)\left(e_{m,3}, e_{m,4}\right), \text{ for all } m \in \mathbb{Z},$$

Consider the orbit algebras

$$A = \widehat{B}/(\nu_{\widehat{B}}), \quad A^{(1)} = \widehat{B}/(\sigma_1 \nu_{\widehat{B}}),$$
$$A^{(2)} = \widehat{B}/(\sigma_2 \nu_{\widehat{B}}), \quad A^{(3)} = \widehat{B}/(\sigma_3 \nu_{\widehat{B}}),$$
$$A^{(4)} = \widehat{B}/(\sigma_4 \nu_{\widehat{B}}).$$

Then $A \cong KQ/I$, $A^{(1)} \cong KQ/I^{(1)}$, $A^{(2)} \cong KQ/I^{(2)}$, $A^{(3)} \cong KQ/I^{(3)}$, $A^{(4)} \cong KQ/I^{(4)}$, where

$Q:$ 

and the ideals $I, I^{(1)}, I^{(2)}, I^{(3)}, I^{(4)}$ are of the form

$$I = \left\langle \begin{array}{c} \beta_1\alpha_1 - \beta_2\alpha_2,\ \beta_2\alpha_2 - \beta_3\alpha_3,\ \beta_3\alpha_3 - \beta_4\alpha_4, \\ \alpha_1\beta_2,\ \alpha_1\beta_3,\ \alpha_1\beta_4,\ \alpha_2\beta_1,\ \alpha_2\beta_3,\ \alpha_2\beta_4, \\ \alpha_3\beta_1,\ \alpha_3\beta_2,\ \alpha_3\beta_4,\ \alpha_4\beta_1,\ \alpha_4\beta_2,\ \alpha_4\beta_3 \end{array} \right\rangle$$

$$I^{(1)} = \left\langle \begin{array}{c} \beta_1\alpha_1 - \beta_2\alpha_2,\ \beta_2\alpha_2 - \beta_3\alpha_3,\ \beta_3\alpha_3 - \beta_4\alpha_4, \\ \alpha_1\beta_1,\ \alpha_1\beta_3,\ \alpha_1\beta_4,\ \alpha_2\beta_2,\ \alpha_2\beta_3,\ \alpha_2\beta_4, \\ \alpha_3\beta_1,\ \alpha_3\beta_2,\ \alpha_3\beta_4,\ \alpha_4\beta_1,\ \alpha_4\beta_2,\ \alpha_4\beta_3 \end{array} \right\rangle$$

$$I^{(2)} = \left\langle \begin{array}{c} \beta_1\alpha_1 - \beta_2\alpha_2,\ \beta_2\alpha_2 - \beta_3\alpha_3,\ \beta_3\alpha_3 - \beta_4\alpha_4, \\ \alpha_1\beta_1,\ \alpha_1\beta_3,\ \alpha_1\beta_4,\ \alpha_2\beta_1,\ \alpha_2\beta_2,\ \alpha_2\beta_4, \\ \alpha_3\beta_2,\ \alpha_3\beta_3,\ \alpha_3\beta_4,\ \alpha_4\beta_1,\ \alpha_4\beta_2,\ \alpha_4\beta_3 \end{array} \right\rangle$$

$$I^{(3)} = \left\langle \begin{array}{c} \beta_1\alpha_1 - \beta_2\alpha_2,\ \beta_2\alpha_2 - \beta_3\alpha_3,\ \beta_3\alpha_3 - \beta_4\alpha_4, \\ \alpha_1\beta_1,\ \alpha_1\beta_3,\ \alpha_1\beta_4,\ \alpha_2\beta_1,\ \alpha_2\beta_2,\ \alpha_2\beta_4, \\ \alpha_3\beta_1,\ \alpha_3\beta_2,\ \alpha_3\beta_3,\ \alpha_4\beta_2,\ \alpha_4\beta_3,\ \alpha_4\beta_4 \end{array} \right\rangle$$

$$I^{(4)} = \left\langle \begin{array}{c} \beta_1\alpha_1 - \beta_2\alpha_2,\ \beta_2\alpha_2 - \beta_3\alpha_3,\ \beta_3\alpha_3 - \beta_4\alpha_4, \\ \alpha_1\beta_1,\ \alpha_1\beta_3,\ \alpha_1\beta_4,\ \alpha_2\beta_2,\ \alpha_2\beta_3,\ \alpha_2\beta_4, \\ \alpha_3\beta_1,\ \alpha_3\beta_2,\ \alpha_3\beta_3,\ \alpha_4\beta_1,\ \alpha_4\beta_2,\ \alpha_4\beta_4 \end{array} \right\rangle.$$

In fact the orbit algebras $\widehat{B}/G$ of $\widehat{B}$ are of the form

$$\mathsf{T}(B)^{(r)} = \widehat{B}/(\nu^r_{\widehat{B}}),\quad \widehat{B}/(\sigma_1\nu^r_{\widehat{B}}),\quad \widehat{B}/(\sigma_2\nu^r_{\widehat{B}}),$$
$$\widehat{B}/(\sigma_3\nu^r_{\widehat{B}}),\quad \widehat{B}/(\sigma_4\nu^r_{\widehat{B}}),$$

for $r \geq 1$.

algebra = basic, connected, finite dimensional K-algebra over a field K

Important selfinjective algebras are socle equivalent to the selfinjective orbit algebras $\widehat{B}/G$, where B is an algebra of finite global dimension, $\widehat{B}$ the repetitive category of B and G is an infinite cyclic group of automorphisms of $\widehat{B}$. Then we may recover the representation theory of $\widehat{B}/G$ from the representation theory of B and the associated derived category $D^b(\operatorname{mod} B) \cong \underline{\operatorname{mod}}\widehat{B}$.

In the theory, an important role is played by the **selfinjective algebras of quasitilted type**.

B **quasitilted** if $B \cong \mathrm{End}_{\mathscr{H}}(T)$ for a hereditary abelian K-category $\mathscr{H}$ and a tilting object T in $\mathscr{H}$.

B is quasitilted $\iff$ $\mathrm{gl.dim}\, B \leq 2$ and every indecomposable module X in $\mathrm{mod}\, B$ satisfies $\mathrm{pd}_B\, X \leq 1$ or $\mathrm{id}_B\, X \leq 1$
(**Happel-Reiten-Smalø**)

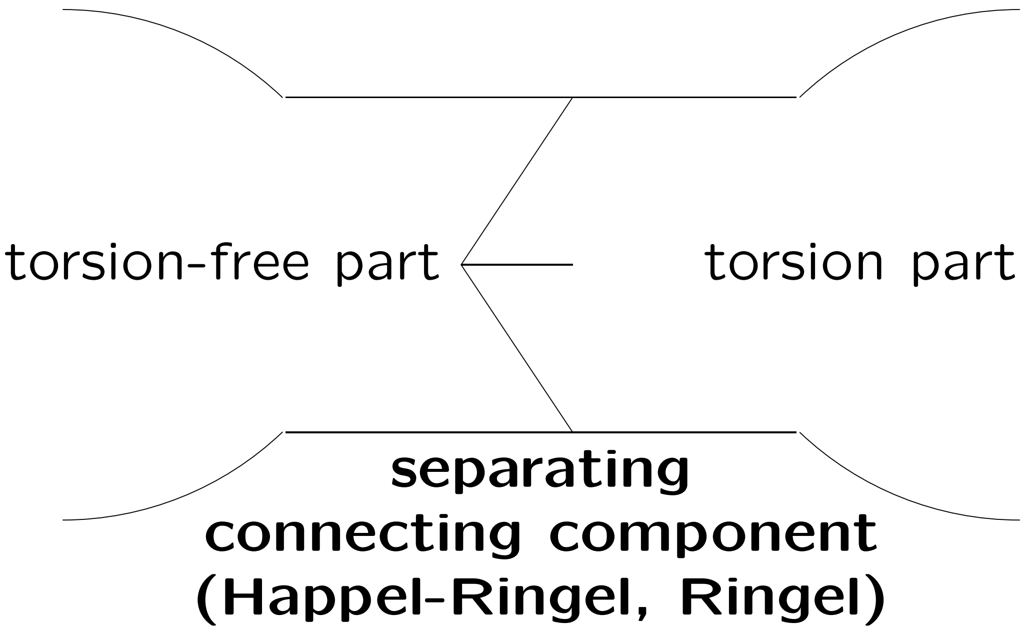
tilted

B quasitilted **Happel, Happel-Reiten**

quasitilted of canonical type

(Tame case: **Skowroński**)

B tilted $\iff$ $\Gamma(\text{mod}\, B)$ of the form



Handy criterion:

Theorem (Liu-Skowroński). *Let B be a connected algebra over a field K. TFAE*

(i) *B is a tilted algebra.*

(ii) *$\Gamma(\text{mod}\, B)$ admits a component $\mathscr{C}$ with a faithful section Σ such that*

$$\text{Hom}_B(X, D\,\text{Tr}\,Y) = 0$$

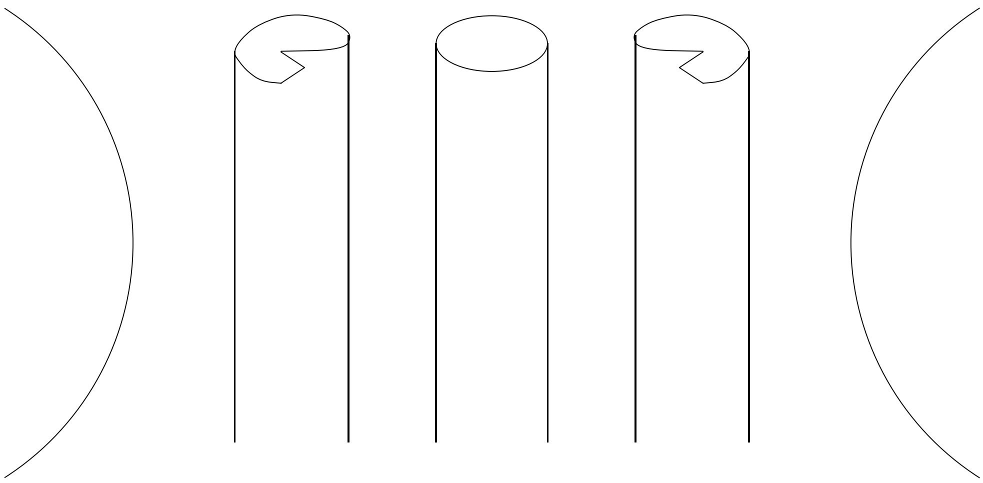
for all modules X, Y of Σ.

Theorem (Lenzing-Skowroński). *Let B be a connected finite dimensional algebra over a field K. TFAE*

(i) *B is a quasitilted algebra of canonical type.*

(ii) *B is a semiregular branch enlargement of a concealed canonical algebra.*

(iii) $\Gamma(\operatorname{mod} B)$ *is of the form*



separating family of semiregular (ray or coray) tubes

Handy criterion:

Theorem (Reiten-Skowroński, Skowroński). *Let B be a connected finite dimensional algebra over a field K. TFAE*

(i) *B is a quasitilted algebra of canonical type.*

(ii) $\Gamma(\operatorname{mod} B)$ *admits a sincere family of pairwise orthogonal semiregular tubes without external short paths.*

B quasitilted algebra

G torsion-free admissible group of automorphisms of the K-category $\widehat{B}$.

$$F : \widehat{B} \longrightarrow \widehat{B}/G = A \qquad \text{Galois covering}$$

$F_\lambda : \operatorname{mod} \widehat{B} \longrightarrow \operatorname{mod} A$ **push-down functor**

Then:

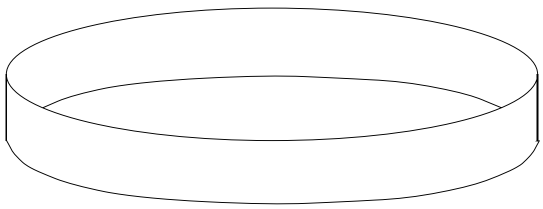
- $G = (\varphi)$, φ strictly positive automorphism of $\widehat{B}$
- $\widehat{B}$ is locally-support finite
- F_λ is dense (application of the **density theorem of Dowbor-Skowroński**)
- $\Gamma(\operatorname{mod} A) = \Gamma(\operatorname{mod} \widehat{B})/G$ (**Gabriel**)

An orbit algebra $\widehat{B}/G$, with B **quasitilted** and G infinite cyclic group of automorphisms of $\widehat{B}$, is called a **selfinjective algebra of quasitilted type**

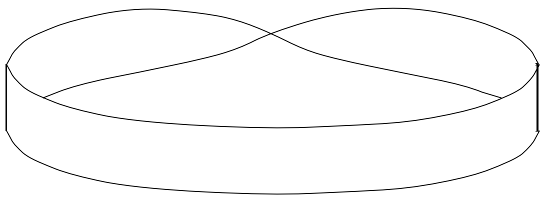
- $\widehat{B}/G$ is **selfinjective of tilted type**, if B is a tilted algebra
- $\widehat{B}/G$ is **selfinjective of Dynkin type**, if B is a tilted algebra of Dynkin type
- $\widehat{B}/G$ is **selfinjective of Euclidean type**, if B is a tilted algebra of Euclidean type
- $\widehat{B}/G$ is **selfinjective of wild tilted type**, if B is a tilted algebra of wild type
- $\widehat{B}/G$ is **selfinjective of tubular type**, if B is a tubular algebra
- $\widehat{B}/G$ is **selfinjective of wild canonical type**, if B is a quasitilted algebra of wild canonical type

Selfinjective algebras of Dynkin type

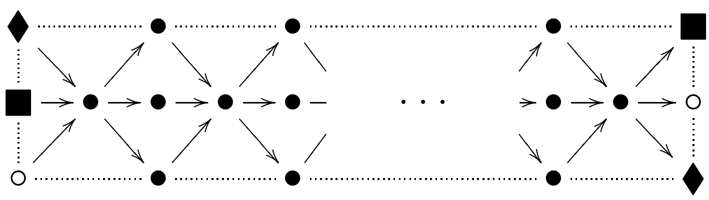
The stable Auslander-Reiten quiver of the form



cylinder



Möbius strip

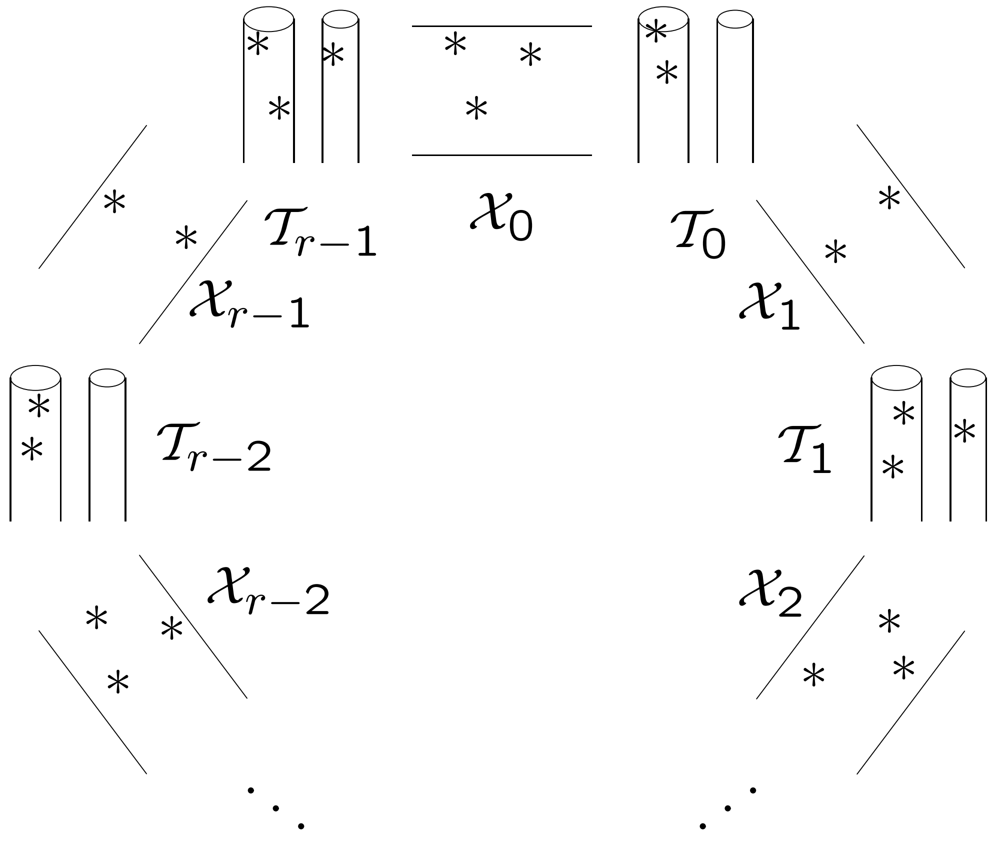


$\mathbb{D}_4$

Riedtmann, **Bretscher-Läser-Reidtmann**, **Hughes-Waschbüsch**, **Waschbüsch**, **Hoshino**, . . .

Selfinjective algebras of Euclidean type

The Auslander-Reiten quiver of the form:



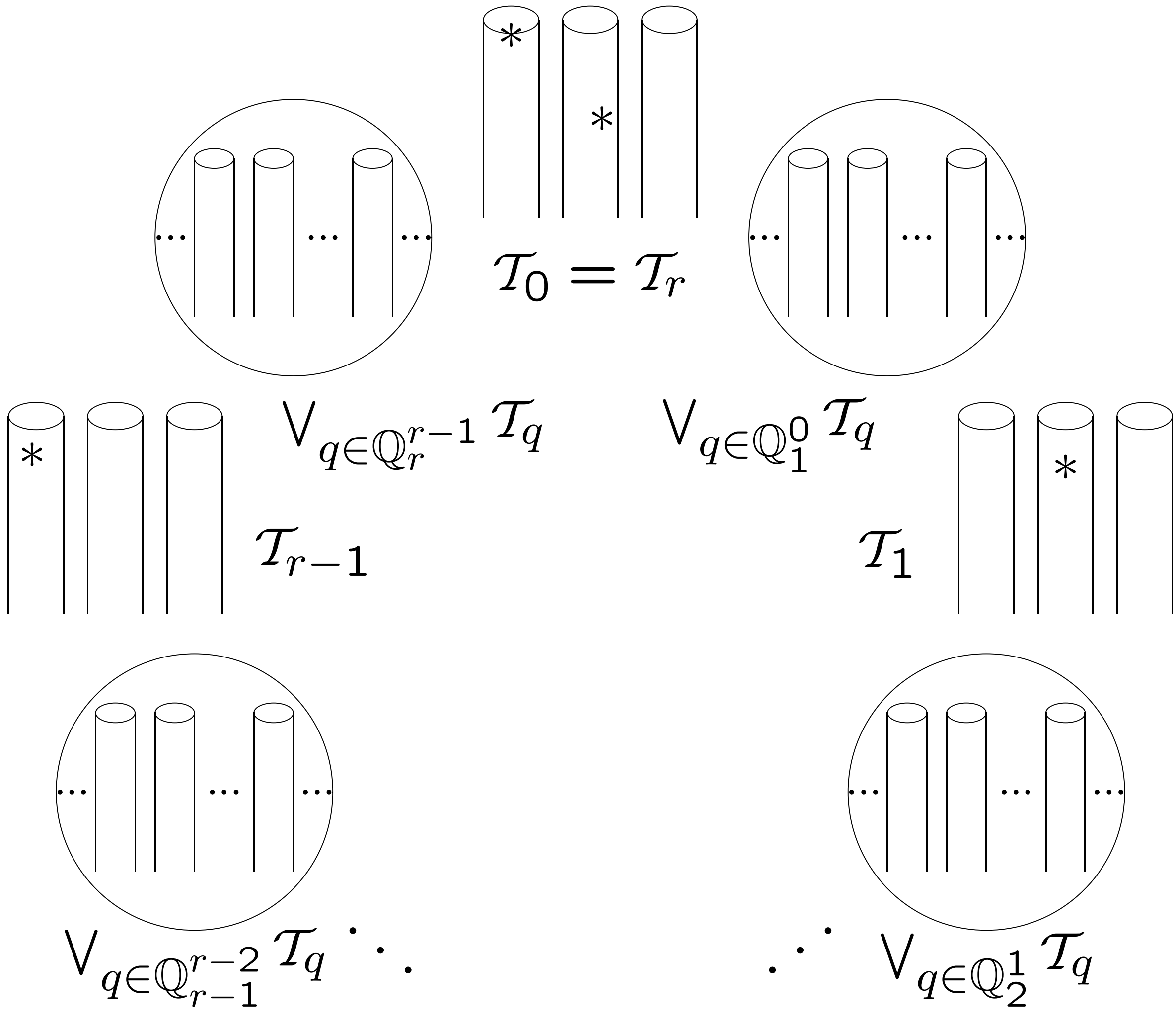
$\mathcal{X}_i^s = \mathbb{Z}\Delta$, Δ Euclidean

$\mathcal{T}_i^s$ = infinite family of stable tubes

Assem-Skowroński,**Assem-Nehring-Skowroński**, **Lenzing-Skowroński**, **Bocian-Skowroński**, ...

Selfinjective algebras of tubular type

The Auslander-Reiten quiver of the form:

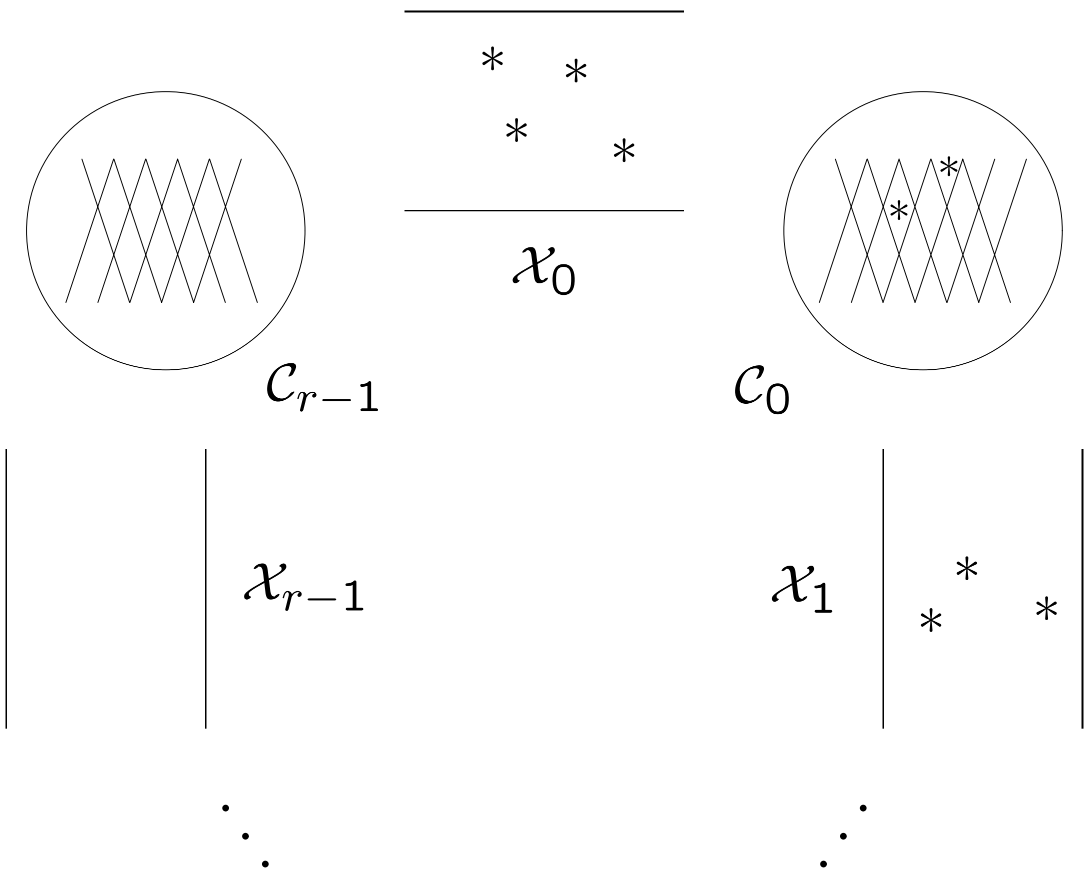


$\mathcal{T}_i^s$ infinite family of stable tubes

Happel-Ringel, **Assem-Skowroński**, **Nehring-Skowroński**, **Lenzing-Skowroński**, **Białkowski-Skowroński**, ...

Selfinjective algebras of wild tilted type

The Auslander-Reiten quiver of the form:



$\mathcal{X}_i^s = \mathbb{Z}\Delta$, Δ wild quiver

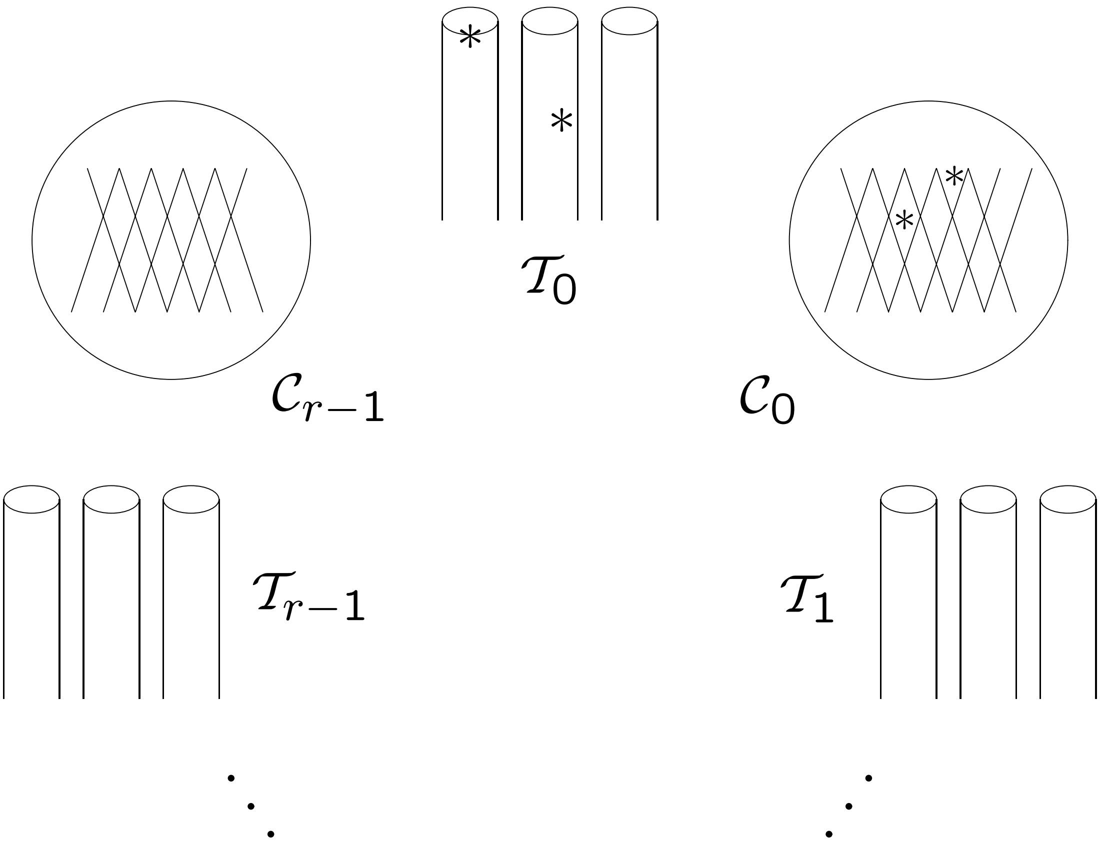
$\mathcal{C}_i^s$ infinite family of components $\mathbb{Z}\mathbb{A}_\infty$

Representation theory:

Erdmann-Kerner-Skowroński

Selfinjective algebras of wild canonical type

The Auslander-Reiten quiver of the form:



$\mathcal{T}_i^s$ infinite family of stable tubes

$\mathcal{C}_i^s$ infinite family of components $\mathbb{Z}\mathbb{A}_\infty$

Representation theory: **Lenzing-Skowroński**

Tame algebras

K algebraically closed field

Λ finite dimensional K-algebra

Λ **tame**: $\forall_{d\geq 1}\exists_{M_1,\ldots,M_{n_d}}$ $K[x]$-Λ-bimodules such that

- M_i free left $K[x]$-modules of finite rank
- all but finitely many isoclasses of indecomposable right Λ-modules of dimension d are of the form $K[x]/(x-\lambda)\otimes_{K[x]} M_i$, $1 \leq i \leq n_d$, $\lambda \in K$

$\mu_\Lambda(d) =$ least number of $K[x]$-Λ-bimodules satisfying the above condition for d

Λ tame $\Longrightarrow$

$$\operatorname{ind}_d \Lambda = \left\{ \begin{array}{c} \textit{finite dis-} \\ \textit{crete set} \end{array} \right\} \bigcup \left\{ \begin{array}{c} \mu_\Lambda(d) \textit{ one-para-} \\ \textit{meter families} \end{array} \right\}$$

Λ is not tame $\stackrel{\textbf{Drozd}}{\Longrightarrow}$ Λ is **wild** (representation theory of Λ comprises the representation theories of all finite dimensional K-algebras)

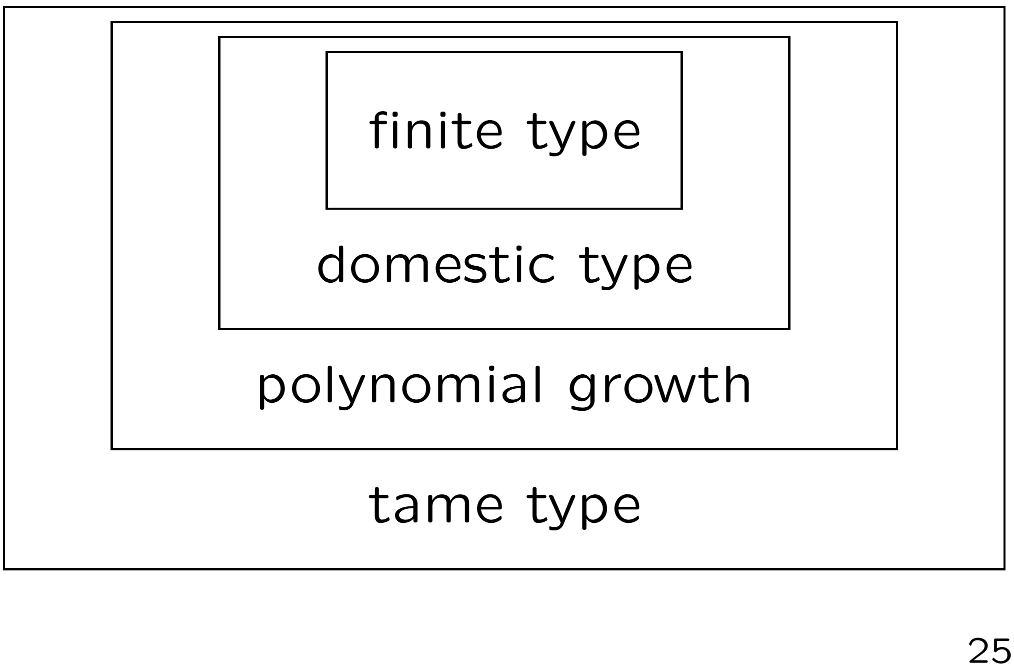
Λ is of finite (representation) type if and only if $\mu_\Lambda(d) = 0$ for all $d \geq 1$ (solution of the **second Brauer-Thrall conjecture**).

Λ tame

Λ is of **polynomial growth**: $\exists_{m \geq 1} \forall_{d \geq 1}\ \mu_\Lambda(d) \leq d^m$

Λ is **domestic (finite growth)**: $\exists_{m \geq 1} \forall_{d \geq 1}\ \mu_\Lambda(d) \leq m$

Hierarchy of tame algebras



Selfinjective algebras of polynomial growth

Theorem (Skowroński). *Let Λ be a non-simple, basic, connected selfinjective algebra over an algebraically closed field K. TFAE*

(i) *Λ is of polynomial growth.*

(ii) *Λ is socle equivalent to a tame selfinjective algebra of quasitilted type.*

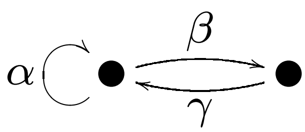
(iii) *Λ is socle equivalent to a selfinjective algebra of Dynkin, Euclidean, or tubular type.*

Moreover, if Λ is of polynomial growth then there exists a unique selfinjective algebra $\bar{\Lambda}$ of quasitilted type such that

- $\dim_K \Lambda = \dim_K \bar{\Lambda}$,
- *Λ and $\bar{\Lambda}$ are socle equivalent,*
- *Λ degenerates to $\bar{\Lambda}$.*

$\bar{\Lambda}$ **standard form** of Λ

Example (Bocian–Skowroński). Let $\Lambda = KQ/I$ where Q is the quiver



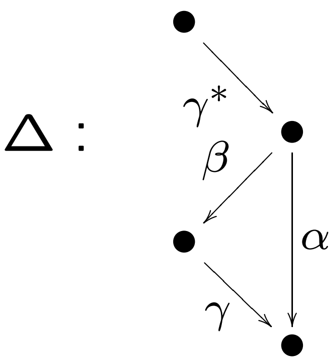
and I ideal of KQ generated by

$$\alpha^2 - \alpha\beta\gamma, \alpha\beta\gamma + \beta\gamma\alpha, \gamma\beta.$$

Then Λ is **socle equivalent** to $\bar{\Lambda} = KQ/\bar{I}$, where $\bar{I}$ ideal of KQ generated by

$$\alpha^2, \alpha\beta\gamma + \beta\gamma\alpha, \gamma\beta.$$

Moreover, $\bar{\Lambda} \cong \widehat{B}/(\varphi)$, where $B = K\Delta/J$ tilted algebra of type $\widetilde{\mathbb{A}}_4$ given by the quiver



J is the ideal of $K\Delta$ generated by $\gamma^*\beta$ and φ is a strictly positive automorphism of $\widehat{B}$ with $\varphi^2 = \varrho\nu_{\widehat{B}}$, for a rigid automorphism ϱ of $\widehat{B}$. Further, $\Lambda \ncong \bar{\Lambda}$, $\dim_K \Lambda = 9 = \dim_K \bar{\Lambda}$ and Λ degenerates to $\bar{\Lambda}$ in the variety of K-algebras of dimension 9.

In fact, Λ is not isomorphic to any orbit algebra $\widehat{R}/G$, R a basic, connected K-algebra.

K arbitrary field

Λ finite dimensional K-algebra

An indecomposable Λ-module M is called a **generic module** if M is of infinite length over Λ but finite length over $\mathrm{End}_\Lambda(M)$ (**endolength** of M), **Crawley-Boevey**

d positive integer

$g_\Lambda(d) =$ the cardinality of isoclasses of generic Λ-modules of endolength d

Λ **generically trivial**: no generic Λ-module

Λ **generically finite**: finite number of isoclasses of generic Λ-module

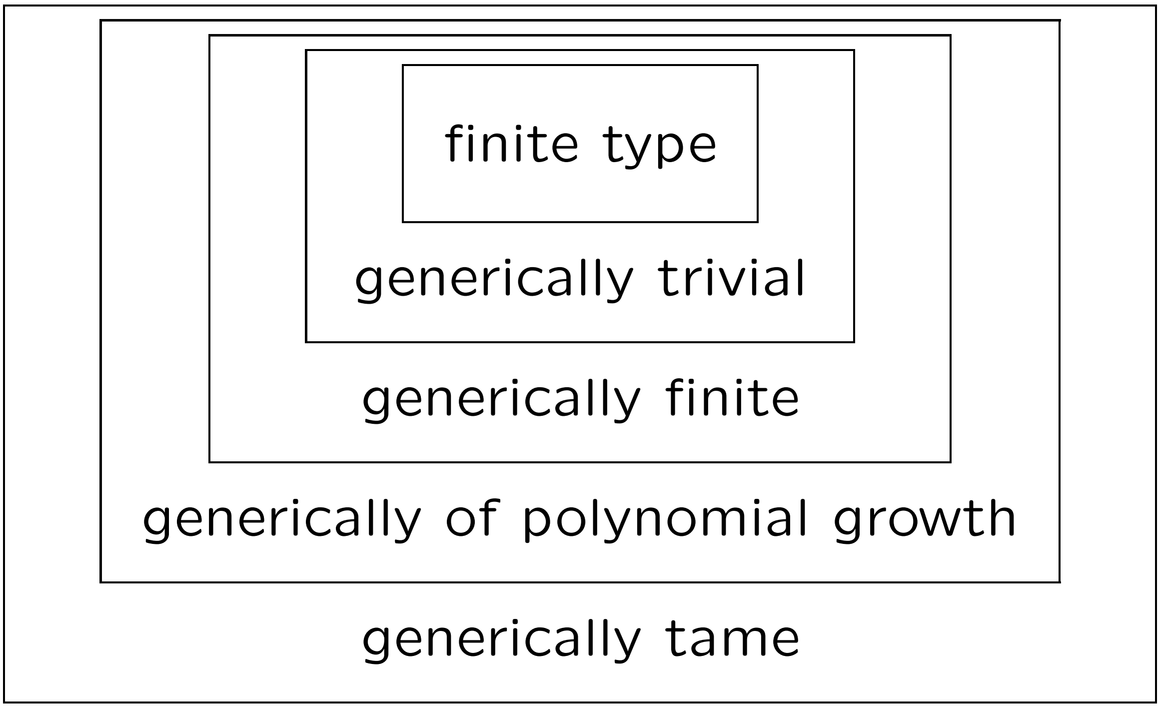
Λ **generically of polynomial growth** : $\underset{\substack{m \geq 1 \\ c \geq 1}}{\exists} \; \underset{d \geq 1}{\forall} \; g_\Lambda(d) \leq c d^m$

Λ **generically tame**: $\underset{d \geq 1}{\forall} \; g_\Lambda(d) < \infty$

Λ **generically wild**: there exists a generic Λ-module M with $\mathrm{End}_\Lambda(M)/\operatorname{rad}\mathrm{End}_\Lambda(M)$ infinite dimensional over the center ($\mathrm{End}_\Lambda(M)$ is not a PI algebra)

Generically tame and wild dichotomy is not clear

Hierarchy of generically tame algebras



For K algebraically closed

finite type $\iff$ generically trivial

domestic type $\iff$ generically finite

polynomial growth $\iff$ generically of polynomial growth

tame type $\iff$ generically tame

Crawley-Boevey

Theorem. *Let A be a selfinjective algebra od quasitilted type. TFAE*

(i) *A is generically tame.*

(ii) *A is generically of polynomial growth.*

(iii) *A is of Dynkin, Euclidean, or tubular type.*

A selfinjective of quasitilted type. Then

A is generically trivial (equivalently of finite type) $\iff$ A is of Dynkin type

A is generically finite $\iff$ A is of Dynkin or Euclidean type

Open problems

Assume A is a basic, connected, finite dimensional selfinjective K-algebra over a field K.

PROBLEM 1. *Assume A is of finite type. Is then A socle equivalent to a selfinjective algebra of Dynkin type?*

PROBLEM 2. *Assume A is generically finite. Is then A socle equivalent to a selfinjective algebra of Dynkin type?*

PROBLEM 3. *Assume A is generically finite but generically nontrivial. Is then A socle equivalent to a selfinjective algebra of Euclidean type?*

PROBLEM 4. *Assume A is generically of polynomial growth but generically infinite. Is then A socle equivalent to a selfinjective algebra of tubular type?*

Λ finite dimensional K-algebra over a field K.

$\operatorname{rad}(\operatorname{mod}\Lambda)$ Jacobson radical of $\operatorname{mod}\Lambda$

$$\operatorname{rad}^{\infty}(\operatorname{mod}\Lambda) = \bigcap_{i=1} \operatorname{rad}^{i}(\operatorname{mod}\Lambda)$$

infinite radical of $\operatorname{mod}\Lambda$

Λ is of finite type $\iff$ $\operatorname{rad}^{\infty}(\operatorname{mod}\Lambda) = 0$ (**Auslander**)

In fact, Λ is of finite type $\iff$ $(\operatorname{rad}^{\infty}(\operatorname{mod}\Lambda))^2 = 0$ (**Coelho-Marcos-Merklen-Skowroński**)

PROBLEM 5. *Describe the selfinjective algebras A with* $\operatorname{rad}^{\infty}(\operatorname{mod} A)$ *nilpotent.*

$$\exists_{m\geq 1} \ (\operatorname{rad}^{\infty}(\operatorname{mod} A))^m = 0$$

PROBLEM 6. *Describe the selfinjective algebras A with* $\operatorname{rad}^{\infty}(\operatorname{mod} A)$ *locally nilpotent.*

$\exists_{m\geq 1} \ (\operatorname{rad}^{\infty}(X, X))^m = 0$ for any indecomposable module X in $\operatorname{mod} A$

A selfinjective algebra of Dynkin or Euclidean type $\Rightarrow$ $\mathrm{rad}^{\infty}(\mathrm{mod}\, A)$ is nilpotent.

A selfinjective algebra of Dynkin, Euclidean or tubular type $\Rightarrow$ $\mathrm{rad}^{\infty}(\mathrm{mod}\, A)$ is locally nilpotent.

Assume A is a basic, connected, finite dimensional selfinjective K-algebra over a field K.

$\mathrm{rad}^{\infty}(\mathrm{mod}\, A)$ nilpotent $\stackrel{?}{\Rightarrow}$ A is socle equivalent to a selfinjective algebra of Dynkin or Euclidean type.

$\mathrm{rad}^{\infty}(\mathrm{mod}\, A)$ locally nilpotent $\stackrel{?}{\Rightarrow}$ A is socle equivalent to a selfinjective algebra of Dynkin, Euclidean or tubular type.

Λ finite dimensional K-algebra over a field K.

A component $\mathscr{C}$ of $\Gamma(\operatorname{mod}\Lambda)$ is called **generalized standard** if $\operatorname{rad}^{\infty}(X, Y) = 0$ for all modules $X, Y \in \mathscr{C}$.

Theorem (Skowroński). *Let Λ be a finite dimensional K-algebra over a field K and $\mathscr{C}$ be a generalized standard component of $\Gamma(\operatorname{mod}\Lambda)$. Then $\mathscr{C}$ is almost periodic (all but finitelly many τ_Λ-orbits in $\mathscr{C}$ are periodic.*

In particular, $\mathscr{C}$ contains at most finitely many indecomposable modules of any fixed dimension d.

Λ of finite type $\Rightarrow$ $\Gamma(\operatorname{mod}\Lambda)$ is generalized standard.

A finite dimensional selfinjective algebra over a field K, $\mathscr{C}$ be a generalized standard component of $\Gamma(\operatorname{mod} A)$. Then the stable part $\mathscr{C}^s$ of $\mathscr{C}$ is of one of the forms:

- $\mathbb{Z}\Delta/G$, Δ Dynkin quiver, G infinite cyclic group of automorphisms of the translation quiver $\mathbb{Z}\Delta$
- $\mathbb{Z}\Delta$, Δ finite, acyclic, valued quiver, not Dynkin
- $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ stable tube of rank r (for some $r \geq 1$)

PROBLEM 7. *Describe the selfinjective algebras A for which the Auslander-Reiten quiver* $\Gamma(\operatorname{mod} A)$ *admits a generalized standard component.*

Positive Galois coverings of selfinjective algebras (Skowroński-Yamagata)

Theorem. *Let A be a basic, connected selfinjective algebra over a field K. TFAE*

(i) $A \cong \widehat{B}/(\varphi\nu_{\widehat{B}})$, *where B is a basic, connected algebra over K and φ is a positive automorphism of $\widehat{B}$.*

(ii) *There exist an ideal I of A and an idempotent e of A such that*

  (1) $r_A(I) = eI$,

  (2) *the canonical algebra epimorphism $eAe \to eAe/eIe$ splits.*

Moreover, in this case, $B \cong A/I \cong eAe/eIe$.

$r_A(I) = \left\{a \in A \,\middle|\, Ia = 0\right\}$ right annihilator of I

$\ell_A(I) = \left\{a \in A \,\middle|\, aI = 0\right\}$ left annihilator of I

In the above theorem, (1) is equivalent to

(1′) $\ell_A(I) = Ie$.

Moreover, $e + I$ is the identity of A/I.

A basic, connected, finite dimensional, self-injective algebra over a field K

An ideal I of A is said to be **deforming** if

- A/I is a triangular algebra,
- $r_A(I) = eI$ for an idempotent e of A.

Then I is an (eAe/eIe)-bimodule

$A \mapsto A[I]$ algebra

$A[I] = (eAe/eIe) \oplus I$ as K-vector space

$$(b, x) \cdot (b', x') = (bb', bx' + xb' + xx')$$

for $b, b' \in eAe/eIe$ and $x, x' \in I$.

$A[I]$ an algebra with the identity $(e, 1 - e)$

$I = \left\{(0, x) \,\middle|\, x \in I\right\}$ ideal of $A[I]$

I is a deforming ideal of $A[I]$

$r_{A[I]}(I) = eI$, $e = (e, 0)$

$A[I]/I = eAe/eIe \cong A/I$

$eA[I]e \longrightarrow eA[I]e/eIe$ splits

In general, the extension

$$0 \longrightarrow I \longrightarrow A[I] \longrightarrow eAe/eIe \longrightarrow 0$$

is not Hochschild extension (in general $I^2 \neq 0$)

Theorem (Skowroński–Yamagata). *Let A be a basic, connected, selfinjective algebra, I a deforming ideal of A. Then the following statements hold:*

- *$A[I]$ is a selfinjective algebra with the same Nakayama permutation as A.*
- *A and $A[I]$ are socle equivalent.*
- *A and $A[I]$ are stable equivalent.*
- *$A \cong A[I]$ if $e_i \neq e_{\nu(i)}$ for any primitive summand e_i of e.*
- *$A[I] \cong \widehat{B}/(\varphi\nu_{\widehat{B}})$ for $B = A/I$ and a positive automorphism φ of $\widehat{B}$.*

Theorem. *Let A be a basic, connected, selfinjective algebra over an algebraically closed field K. TFAE*

- *A admits a deforming ideal I.*
- *$A \cong \widehat{B}/(\varphi\nu_{\widehat{B}})$, B a triangular algebra, φ a positive automorphism of $\widehat{B}$.*

A selfinjective algebra over an algebraically closed field K, I deforming ideal of A and e an idempotent of A with $r_A(I) = eI$, then

$$H^2(eAe/eIe, eIe) = 0.$$

In particular, the canonical algebra epimorphism $eAe \longrightarrow eAe/eIe$ splits

$$0 \longrightarrow eIe \longrightarrow eAe \longrightarrow eAe/eIe \longrightarrow 0$$

is Hochschild extension because $(eIe)^2 = 0$.

Example. Let $K \subseteq L$ finite field extension with $H^2(L, L) \neq 0$, L considered as K-algebra, $\alpha : L \times L \to L$ 2-cocycle defining a nonsplittable Hochschild extension

$$0 \longrightarrow L \longrightarrow E \longrightarrow L \longrightarrow 0$$

[For example, $K = \mathbb{Z}_2(u)$ field of rational functions in one variable u over $\mathbb{Z}_2$, $L = K[X]/(X^2 - u)$, x residue class of X, $\alpha : L \times L \to L$ given by $\alpha(x^l, x^m) = x^{l+m}$ for $l, m \in \{0, 1\}$.]

Let Q be a finite acyclic quiver and $H = LQ$ the path algebra of Q over L.

H is a hereditary K-algebra

Let Q_0 the set of vertices of Q, $e_i (i \in Q_0)$ corresponding set of primitive idempotents of $H = LQ$, $e_i^* (i \in Q_0)$ dual elements in

$$D(H) = \mathrm{Hom}_L(H, L) = \mathrm{Hom}_K(L, K)$$

Consider the 2-cocycle $\widehat{\alpha} : H \times H \to D(H)$ of the form

$$\widehat{\alpha}(a, b) = \sum_{i \in Q_0} \alpha(a_i, b_i) e_i^*$$

with $a_i = e_i a e_i$, $b_i = e_i b e_i$, $i \in Q_0$, for all $a, b \in LQ$.

Then we have a nonsplittable Hochschild extension

$$0 \longrightarrow D(H) \longrightarrow A \longrightarrow H \longrightarrow 0$$

$I = D(H)$ ideal of A with $I^2 = 0$,

$A = H \oplus D(H)$ as K-vector space

$$(a, x) \cdot (b, y) = (ab,\, ay + xb + \widehat{\alpha}(a, b))$$

for $a, b \in H$, $x, y \in D(H)$.

$(e_i, -\alpha(1,1)e_i^*)$, $i \in Q_0$, complete set of pairwise orthogonal primitive idempotents of A

$1_A = (1, -\alpha(1,1) \sum_{i \in Q_0} e_i^*)$ identity of A

$e = 1_A$ residual identity of $H = A/I$

The quiver Q_H of H is acyclic

$r_A(I) = I = eI$, $\ell_A(I) = I = Ie$

Hence I is a **deforming ideal** of A

$A[I] \cong H \ltimes D(H) = \widehat{H}/(\nu_{\widehat{H}})$ trivial extension of H by $D(H)$

A and $A[I]$ are **not isomorphic**

A and $A[I]$ are **socle and stably equivalent**

$A[I]$ symmetric, and A is not symmetric

Hence, A and $A[I]$ are **not derived equivalent**

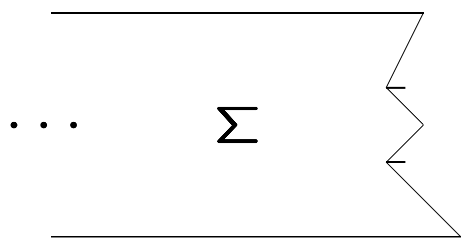
Theorem (Skowroński-Yamagata). *Let A be a selfinjective algebra over a field K. TFAE*

(i) A *is socle equivalent to a selfinjective algebra* $\widehat{B}/(\varphi\nu_{\widehat{B}})$*, where* B *is a tilted algebra not of Dynkin type and* φ *is a positive automorphism of* $\widehat{B}$*.*

(ii) $\Gamma(\text{mod}\, A)$ *admits a generalized standard acyclic full translation subquiver* Σ *which is closed under predecessors in* $\Gamma(\text{mod}\, A)$*.*

(iii) $\Gamma(\text{mod}\, A)$ *admits a generalized standard acyclic full translation subquiver* Ω *which is closed under successors in* $\Gamma(\text{mod}\, A)$*.*

Moreover, if K *is algebraically closed, we may replace in* (i) *"socle equivalent" by "isomorphic".*



I annihilator of Σ in A

$B = A/I$ a tilted algebra not of Dynkin type

I deforming ideal of A

Theorem (Skowroński-Yamagata). *Let A be a selfinjective algebra over a field K. TFAE*

(i) *A is socle equivalent to a selfinjective algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$, B is a tilted algebra and φ is a positive automorphism of $\widehat{B}$.*

(ii) *A is stably equivalent to a selfinjective algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$, B is a tilted algebra and φ is a positive automorphism of $\widehat{B}$.*

Corollary. *The class of selfinjective algebras over an algebraically closed field K of the form $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where B is a tilted algebra and φ is a positive automorphism of $\widehat{B}$ is closed under stable (derived) equivalences.*

For selfinjective algebras of quasitilted type, we have the following

Theorem (Kerner-Skowroński-Yamagata). *The class of selfinjective algebras over a field* K *of the form* $\widehat{B}/(\varphi\nu_{\widehat{B}})$*, where* B *is a quasitilted algebra and* φ *is a strictly positive automorphism of* $\widehat{B}$*, is closed under stable (derived) equivalences.*

Theorem (Skowroński-Yamagata). *Let A be a selfinjective algebra of infinite type over a field K. TFAE*

(i) *Every component of* $\Gamma(\text{mod}\, A)$ *is generalized standard.*

(ii) $A \cong \widehat{B}/(\varphi\nu_{\widehat{B}})$, *where B is a tilted algebra of Euclidean type or a tubular algebra, and φ is a strictly positive automorphism of $\widehat{B}$.*

The proof applies

Theorem (Skowroński). *Let A be a basic, connected representation-infinite algebra over a field K. TFAE*

(i) *A is a concealed algebra of Euclidean type.*

(ii) *A satisfies two conditions:*

  (a) *A is of infinite type but A/I is of finite type for any nonzero ideal I of A.*

  (b) $\text{rad}^{\infty}(X, X) = 0$ *for any indecomposable finite dimensional A-module X.*

Theorem (Skowroński). *Let* Λ *be a basic algebra over a field* K *and* r *a positive integer. Then there exist a basic connected symmetric algebra* A *over* K *such that*

(i) $\Gamma(\text{mod}\, A)$ *admits a sincere generalized standard stable tube of rank* r*.*

(ii) Λ *is a factor algebra of* A*.*

Λ basic finite dimensional K-algebra

M arbitrary faithful module in $\operatorname{mod}\Lambda$

(for example $M = \Lambda \oplus X$ for a module X in $\operatorname{mod}\Lambda$)

For $r = 1$, take

$$B = \begin{bmatrix} K & M & K^2 \\ 0 & \Lambda & D(M) \\ 0 & 0 & K \end{bmatrix}$$

multiplication given by

K-Λ-bimodule structure of M

Λ-K-bimodule structure of $D(M) = \operatorname{Hom}_K(M, K)$

canonical K-K-bimodule structure of K^2

$\varphi : M \otimes_\Lambda D(M) \longrightarrow K$ evaluation map

$\varphi(m \otimes f) = f(m)$, for $m \in M$, $f \in D(M)$

For $r \geq 2$, consider the algebra $R = T_{r-1}(K)$ of $(r-1) \times (r-1)$, upper triangular matrices over K, $T_1(K) = K$, N unique indecomposable projective-injective R-module.

Take

$$B = \begin{bmatrix} K & M \oplus N & K^2 \\ 0 & \Lambda \times R & D(M) \oplus D(N) \\ 0 & 0 & K \end{bmatrix}$$

multiplication given by

K-$(\Lambda \times R)$-bimodule structure of $M \oplus N$,

$(\Lambda \times R)$-K-bimodule structure of $D(M) \oplus D(N)$,

canonical K-K-bimodule structure of K^2

$\theta : (M \oplus N) \otimes_{\Lambda \times R} (D(M) \oplus D(N)) \longrightarrow K^2$
induced by evaluation maps
$\varphi : M \otimes_{\Lambda} D(M) \to K$, $\psi : N \otimes_R D(N) \to K$

B a **generalized canonical algebra**

$A = \mathrm{T}(B) = B \ltimes D(B)$ symmetric algebra

$\Gamma(\operatorname{mod} A)$ admits a sincere generalized standard stable tube of rank r

Theorem (Białkowski-Skowroński-Yamagata). *Let A be a symmetric algebra over a field K such that* $\Gamma(\text{mod}\, A)$ *admits a generalized standard stable tube. Then the Cartan matrix C_A of A is singular.*

There exist selfinjective algebras A such that

- C_A is nonsingular,
- $\Gamma(\text{mod}\, A)$ admits a generalized standard stable tube