Disorder effects on phase transitions

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As an example of the Imry- Ma phenomenon, the famed discontinuity of the magnetization in the $2D$ Ising model is unstable to the addition of quenched random magnetic field of uniform variance, even if that is small.

For $O(n)$ models with rotationally invariant couplings, the rounding effect extends to dimensions $d \leq 4$, provided also the quenched disorder's distribution is invariant under the spin rotation.

The talk will center on the decay rate of the effect of boundary conditions on the magnetization, as function of the distance to the boundary.

The most recent result is a power-law upper bound on the corresponding quenched correlation function, which is valid for all field strengths and at all temperatures, including $T = 0$. The analysis proceeds through a better quantified variant of the Aiz.-Wehr proof of the Imry-Ma rounding effect.

At strong disorder, and/or high temperatures, the quenched correlations decay exponentially. The existent results still leave open the question of a possible phase transition between exponential and power-law decay of the influence function, reached by varying the disorder strength at very low temperatures.

The talk is based on a joint work with Ron Peled (TAU).

Collaborators on related past works: Jan Wehr, Joel Lebowitz, Rafael Greenblatt.

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Quenched disorder, and its effects on phase transitions

Models modified by quenched disorder (denoted η, of strength ε)

1) Random Field Ising Model (RFIM): $\sigma_v = \pm 1$ for $v \in \mathbb{Z}^d$

$$H_\eta(\sigma) := -J \sum_{\substack{\{u,v\}\subset\mathbb{Z}^d \\ u\sim v}} \sigma_u\sigma_v - \sum_{v\in\mathbb{Z}^d} (h + \varepsilon\,\eta_v)\sigma_v \tag{1}$$

with $\{\eta_v\}_{v\in\mathbb{Z}^d}$ independent random variables, e.g. iid normal gaussian, $N(0,1)$.

2) Random Field O(N) Model
$\{\sigma_v, h, \eta_v\} \Rightarrow \{\vec{\sigma}_v, \vec{h}, \vec{\eta}_v\}$, N component vectors; $\sigma_u\sigma_v \Rightarrow \vec{\sigma}_u \cdot \vec{\sigma}_v$, etc.
with $\vec{\eta}_v$ given by independent random variables of rotation invariant distribution.

3) Q-state Potts model ($\sigma_v \in \{1, ..., Q\}$) with randomized couplings

$$H_\eta(\sigma) = - \sum_{\substack{\{u,v\}\subset\mathbb{Z}^2 \\ u\sim v}} (J + \varepsilon\,\eta_{x,y})\,\mathbb{1}[\sigma_x = \sigma_y] - \sum_{x\in\mathbb{Z}^2} h\,\mathbb{1}[\sigma_x = 1] \tag{2}$$

with $\eta_{x,y}$ uniformly distributed over $[-1, 1]$ (w.r.t. Lebesgue measure).

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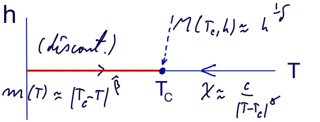
Quenched Disorder's effects on phase transitions

Without the disorder, in d-dimensions the above models exhibit 1^{st} order phase transitions, provided:

Ising	$d > 1$
$O(N)$	$d > 2$
Q-state Potts	$d = 2, \quad Q > 4$



Ising order parameter $m(T)$ defined as

$$m(T) = \lim_{h \downarrow 0} M(T, h)$$

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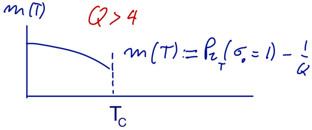

$\widehat{\beta}$, γ, δ – critical exponents

Initial questions:

Q1) Does the first order transition persist under the quenched disorder?

Q2) If so: does the disorder affect the critical exponents?
(Q2 will not be discussed here - Harris criterion & all that.)

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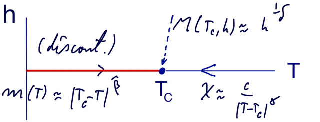

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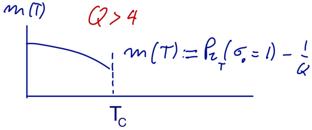

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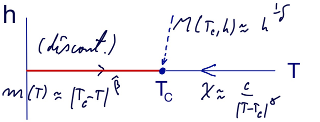

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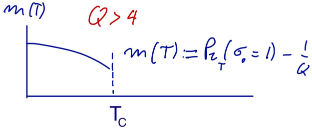

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Concerning $Q1$:

The *Imry-Ma argument* won over *dimensional reduction*

I) Imry-Ma prediction:

$$1^{st} \textit{ order discontinuity persists iff } \begin{cases} d > 2 & \text{discrete systems} \quad (L^{d/2} \geq L^{d-1}) \\ d > 4 & \text{cont. symm.} \quad (L^{d/2} \geq L^{d-2}) \end{cases}$$

Y. Imry and S.K. Ma, PRL **35** (1975).

II) An alternative "dimensional reduction proposal" (disproved for Ising model):

$$d_{lc}(disord.) = d_{lc}(homog.) - 2\,.$$

and also in terms of the critical exponents

disordered systems in dim. d $\approx$ *homogen. systems in dim. $(d-2)$.*

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*Q*1 is equivalent to: does the free energy $\mathcal{F}(h, \beta, \varepsilon)$ retain its kink singularity in h under quenched disorder, i.e. for all $\varepsilon > 0$?

Quenched free energy: For random fields $\eta = (\eta_v)$ of finite variance, forming a translation invariant and ergodic process: $\forall \beta \in [0, \infty]$ there is a full measure set of configurations η for which

$$\mathcal{F}(h, \beta, \epsilon) := \lim_{L \to \infty} \frac{-1}{\beta\, |\Lambda_L|} \log Z_{\Lambda_L, B}(h, \beta, \epsilon; \eta) \tag{3}$$

exists and its value is independent of η and of the boundary conditions B.

Furthermore

i. $\mathcal{F}(h, \beta, \epsilon)$ is concave as a function of h.

ii. $\lim_{\beta \to \infty} \mathcal{F}(h, \beta, \epsilon)$ gives the (a.s.) ground state energy *density*
($\beta \to \infty$ and $L \to \infty$ are interchangeable for $\mathcal{F}$.)

iii. Uniqueness of the limit does not extend to uniqueness of the Gibbs states.
However: for any (β, h, ε) at which $\mathcal{F}$ is differentiable in h

$$\lim_{L \to \infty} \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \left\langle \frac{\partial H}{\partial h_x} \right\rangle^{h}_{\Lambda_L, B} (\eta) = \frac{\partial \mathcal{F}}{\partial h} \tag{4}$$

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Rigorous results concerning Q1 - (η of cont. distr. $\mathbb{E}(\eta^4) < \infty$)

Thm 1 (1st order phase transition persists in 3D)
In $d = 3$ dimensions, at sufficiently small ε the RFIM continues to display a 1st order transition in h, both at $T = 0$ and at small enough temperatures.

T=0: J.Z. Imbrie, PRL '84 / CMP '85
T> 0: J. Bricmont, A. Kupiainen, PRL '87 / CMP '88

Thm 2 (Rounding of the phase transition in 2D)
In $d = 2$ dimensions, at any $\varepsilon > 0$: the RFIM has almost surely a unique ground state, and a unique Gibbs states at any (β, h).

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Thm 2' (A more general statement [AW])
In $d = 2$ dimensions, at any $\varepsilon > 0$ the free energy is differentiable in the parameter to which disorder was added (such as h in RFIM).

- For systems with continuous symmetry Thm 2' extends to in $d \leq 4$ dims (free energy is diff. at $\vec{h} = \vec{0}$, provided also the distribution of $\vec{\eta}$ is rot. inv.)

- Thm 2' holds also for Quantum Systems.

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A handy tool for the non-perturbative argument ([AW])

For the RFIM at $T = 0$ and $h = 0$, and volumes $\Lambda(L) = [-L, L]^2$, consider:

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G_L(\eta) \ :=\ & \left[\min_\sigma H^+_{\Lambda(3L),\eta}(\sigma) - \min_\sigma H^-_{\Lambda(3L),\eta}(\sigma)\right] \\
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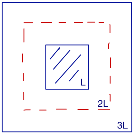

where $\pm$ denotes the boundary conditions on the outer box,
and $R_L\eta$ is the field obtained by setting $\eta_x = 0$ within the inner box.

This quantity obeys:

1) the uniform bound: $|G_L(\eta)| \ \leq\ 4J\,|\partial\Lambda(2L)| = \text{Const.}L^{d-1} \quad (\forall \eta)$

2) for each $x \in \Lambda(L)$:

$$\frac{\partial}{\partial \eta_x} G_L(\eta) \ =\ -\left[\widehat{\sigma}^+_x(\eta) - \widehat{\sigma}^-_x(\eta)\right] \ =\ -2\mathbb{1}\left[\widehat{\sigma}^+_x(\eta) \neq \widehat{\sigma}^-_x(\eta)\right]$$

From (2) (+ ergodicity) one can deduce the anti-concentration bound:

if $\Pr\left(\sigma^+_0 \neq \widehat{\sigma}^-_0\right) = m(\varepsilon) \neq 0$ then $\mathcal{D} - \lim_{L\to\infty} G_L \ =\ \theta(m(\varepsilon))\, L^{d/2}\, N(0,1)$

i.e., $G_L(\eta) \approx \theta(m) \times$ norm. Gaussian var..

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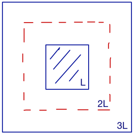

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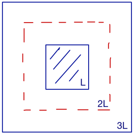

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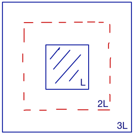
A handy tool for the non-perturbative argument ([AW])

For the RFIM at $T = 0$ and $h = 0$, and volumes $\Lambda(L) = [-L, L]^2$, consider:

$$G_L(\eta) := \left[\min_\sigma H^+_{\Lambda(3L),\eta}(\sigma) - \min_\sigma H^-_{\Lambda(3L),\eta}(\sigma)\right] - \left[\min_\sigma H^+_{\Lambda(3L),R_L\eta}(\sigma) - \min_\sigma H^-_{\Lambda(3L),R_L\eta}(\sigma)\right]$$



where $\pm$ denotes the boundary conditions on the outer box,
and $R_L\eta$ is the field obtained by setting $\eta_x = 0$ within the inner box.

This quantity obeys:

1) the uniform bound: $\boxed{|G_L(\eta)| \leq 4J\,|\partial\Lambda(2L)| = \text{Const.} L^{d-1} \quad (\forall \eta)}$

2) for each $x \in \Lambda(L)$:

$$\frac{\partial}{\partial \eta_x} G_L(\eta) = -\left[\widehat{\sigma}^+_x(\eta) - \widehat{\sigma}^-_x(\eta)\right] = -2\mathbb{1}\left[\widehat{\sigma}^+_x(\eta) \neq \widehat{\sigma}^-_x(\eta)\right]$$

From (2) (+ ergodicity) one can deduce the anti-concentration bound:

if $\Pr\left(\sigma^+_0 \neq \widehat{\sigma}^-_0\right) = m(\varepsilon) \neq 0$ then $\boxed{\mathcal{D} - \lim_{L\to\infty} G_L = \theta(m(\varepsilon))\, L^{d/2}\, N(0,1)}$

i.e., $G_L(\eta) \approx \theta(m) \times$ norm. Gaussian var..

For $d = 2$ that's a contradiction $\Longrightarrow$ $\boxed{m(\varepsilon) = 0}$.

Next question – influence percolation decay rate

Focusing now on the critical dimensions $d = 2$, and $d = 4$ (for the case of continuous symmetry), we have the following *influence percolation* question

Q3 : At what rate does the unique (!) ground state localize, i.e. decouple from the boundary conditions and disorder, at distance L?

More explicilty, let

$$M_L(\varepsilon) := \frac{1}{2}\,\mathrm{Av}\left(\sigma_0^{\Lambda(L),+} - \widehat{\sigma}_0^{\Lambda(L),-}\right) \overset{\text{Ising}}{=} \Pr\left(\sigma_0^{\Lambda(L),+} \neq \widehat{\sigma}_0^{\Lambda(L),-}\right) .$$

A percolation argument $\Longrightarrow$ in any dimension at strong disorder,

$$\boxed{M_L(\varepsilon) \leq C(\varepsilon) e^{-\mu(\varepsilon) L} \qquad (\text{with } \mu(\varepsilon) > 0 \text{ for } \varepsilon > \varepsilon_d).}$$

(cf. A,-Wehr, A.-Peled, Camia-Jiang-Newman arXiv 2018)

Does this persists to weak disorder, or is there a transition to a phase with a slower decay (still at $T = 0$)? And how slow can that decay be?

The first question was considered early on in

B. Derida and Y. Shnidman,"Possible line of critical points for [RFIM] in dimension 2", J. Phys. Lett. '84.

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Correlation function bounds

Related, but rather weak, recent bound

S. Chatterjee "On the decay of correlations in the [RFIM]", CMP '2018

where it is shown that $(\forall\varepsilon)$: $M_L \leq C(\varepsilon)/\sqrt{\log\log L}$

A better bound was derived through a significantly improved version of the argument outlined above (jointly with R. Peled, arXiv 2018):

Theorem: for any $T, \varepsilon \geq 0$: $M_L \leq C(\varepsilon)/L^{\alpha(\varepsilon)}$

Our current guess: the actual behavior may well be exponential decay $\forall\varepsilon > 0$. However, at very large correlation length, e.g. $\ell(\varepsilon) \approx e^{-C/\varepsilon^2}$.

Open Problem 1: is there a phase transition at which the behavior changes?

• The arguments used above were not (yet ?) extended to the case of continuous symmetry breaking under quenched disorder.

Open Problem 2: Is there a K-T like line of critical points in $4D$, at $T = 0$? (This would also be in line with the dimensional reduction picture...)

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A tempting, but potentially misleading picture - Mandelbrot percolation

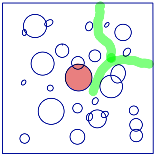
Of possible relevance for an intuitive picture are the scale-invariant *swiss cheese* and the related Mandelbrot's "canonical curdling" models:

J.T. Chayes, L. Chayes, R. Durrett, PTRF '88.

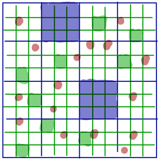
- Consider the Poisson process of spheres $B(x, r)$ in $\mathbb{R}^\nu$ with density

$$\lambda \frac{dr\, d^\nu x}{r^{1+\nu}} \quad \text{over } [0,1] \times \mathbb{R}^\nu \qquad (\text{with } \lambda = e^{-C/\varepsilon^2})$$

The cheese is the set left uncovered. For $\varepsilon < \varepsilon_0$ the set is not empty, though of zero Lebesgue measure. Its dimension, $\dim_{\mathcal{H}}(\text{cheese})$, increases as $\varepsilon \downarrow 0$, and at small enough ε it percolates



The picture may initially suggests that at weak enough disorder the influence percolation may decay by a power law.
But it also leaves room for improvement.



Let me end by repeating:
Open Problem 2: is there a $T = 0$ phase transition in random field $O(n)$ models in $d = 4$ (or less) dimensions?

Thank you for your attention.