Asymptotic properties of actions on rooted trees

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$T = T_d$ - infinite $d$ regular rooted tree

$V(T_d) \cong \{0, \ldots, d-1\}$.

Consider $G < \text{Aut}(T)$, $G = \langle S \rangle$ finitely generated, acting transitively on each level $X_n$ of the tree.

$\Gamma_n = \text{Sch}(G, X_n, S) = \text{Sch}(G, \text{Stab}_G(v), S)$ for any $v \in X_n$

$n$-th Schreier graph.

$\text{Vert}(\Gamma_n) = X_n$; \hspace{1em} $\text{Edges}(\Gamma_n) = \{(v, s(v)) \mid s \in S \cup S^{-1}\}$.

$G$ acts on the boundary $\partial T \cong \{\xi = x_0x_1x_2 \ldots \}$ by homeomorphisms preserving the uniform measure $\lambda$ on $\partial T$. For every $\xi \in \partial T$, consider

$\Gamma_\xi = \text{Sch}(G, \text{Stab}_G(\xi), S)$ the (infinite) orbital Schreier graph.
\[ a = (b, id) \epsilon \]
\[ b = (a, id) \epsilon \]
\( \text{Aut}(T) = \text{Aut}(T) \vartriangleleft \text{Sym}_d \); \  \text{Aut}(T) \ni g = \tau_g(g|_0, \ldots g|_{d-1}) \)

where \( \tau_g \in \text{Sym}_d \) and \( g|_0, \ldots g|_{d-1} \) are restrictions of \( g \) on the subtrees rooted in the vertices of the first level.

A finitely generated subgroup \( G < \text{Aut}(T) \) is **self-similar** if \( g|_v \in G \) for all \( g \in G \) and all \( v \in V(T) \).

If the action is self-similar the Schreier graphs \( \{ \Gamma_n \} \) form a family of graph coverings.

The union \( U_{\xi \in \partial T} \Gamma_\xi \) (with profinite topology) is the inverse limit of the projective sequence \( \{ \Gamma_n \}_n \).
$\mathcal{G}_* = \text{the compact space of (rooted isomorphism classes of) rooted connected graphs of bounded degree, with pointed Gromov-Hausdorff convergence: } (\Gamma_n, r_n) \rightarrow (\Gamma, r) \text{ as } n \rightarrow \infty, \text{ iff for every } k, \exists N \text{ such that } \forall n > N, \text{ the ball } B(r, k) \text{ in } \Gamma \text{ is isomorphic to the ball } B(r_n, k) \text{ in } \Gamma_n.$

$Stab_G(\xi) = \bigcap_{n \in \mathbb{N}} Stab_G(\xi_n)$

$\xi_n$ denotes the prefix of $\xi$ of length $n$.

$(\Gamma_n, \xi_n) \rightarrow (\Gamma_\xi, \xi)$ as $n \rightarrow \infty$, in the space $\mathcal{G}_*$. 

$\xi = 1001 \ldots$
A finitely generated $G < \text{Aut}(T)$ is self-similar iff it is generated by the states of an invertible automaton

$$A=(S,X,\mu,\nu)$$

$\mu: S \times X \rightarrow S$

$\nu: S \times X \rightarrow X$

$$\mu(s, xw) = \mu(\mu(s, x), w),$$

$$\nu(s, xw) = \nu(s, x)\nu(\mu(s, x), w)$$

The automaton generating the Basilica group.

Let $A_s$ be the initial automaton determined by $A$ and $s \in S$. Then $G=\langle A_s \mid s \in S \rangle$. 
A self-similar group $G \triangleleft \text{Aut}(T)$ is contracting iff $\exists$ a finite $N \subset G$ s.t. $\forall g \in G$, $g|_v \in N$ for all $v$ long enough. Finite Schreier graphs of a contracting action provide an approximating sequence for a compact space called Limit Space of $G$ (Nekrashevych).

Draw the edges of finite Schreier graph $\Gamma_n$ on the vertices of each level in the left Cayley graph of $X^*$. The resulting infinite graph is Gromov hyperbolic (first such example: Sierpinski graph, Kaimanovich ‘03).

The hyperbolic boundary of this graph = limit space of $G$ (independent of the generating set, up to homeomorphism).
Example 1. Grigorchuk’s group
\[ G = \langle a, b, c, d \rangle \quad a = \varepsilon(id, id); \quad b = e(a, c)e; \quad c = e(a, d); \quad d = e(1, b) \]

Limit Space of \( G \) is an interval
Example 2.
Basilica Group
(Grigorchuk, Zuk)

\[ B = \langle a, b \rangle \text{ with } a = e(b, \text{id}); \ b = \varepsilon(a, \text{id}) \]
\( B = \text{IMG} \left( z^2 - 1 \right) \)

\[ \Gamma_6 \]

\( n \to \infty \) ... Limit space of B is the Julia set of \( z^2 - 1 \).
Example 3. Hanoi Towers Group $H^{(3)}$.
(Grigorchuk, Sunic)

$H^{(3)} = \langle a,b,c \rangle$

The Hanoi Towers game on three pegs.

Given $n$ disks of different sizes, the game consists in taking all $n$ disks from a peg to another by moving each disk so that at each step we have an allowed configuration. A configuration is allowed if no disk is placed on top of a smaller disk. Words of length $n$ in the alphabet $\{0,1,2\}$ encode the configurations of $n$ disks on three pegs.

Set

$a :=$ moving a disk between peg 0 and 1,
$b :=$ moving a disk between peg 0 and 2,
$c :=$ moving a disk between peg 1 and 2.
The orbit graph of $H^{(3)}$ at level 3
The limit space of the Hanoi towers group $H^{(3)}$ is the Sierpinski gasket
Problem: Study sequences of Schreier graphs $\Gamma_n$ and their various limits

I. First area of application: spectral computations.

$\Gamma_n \rightarrow \Gamma_\xi$ in the space $\mathcal{G}_*$ of rooted graphs $\implies$

$\text{Sp}(\Gamma_n) \rightarrow \text{Sp}(\Gamma_\xi)$.

Example 1: Computation of the spectrum on Cayley graphs. E.g. lamplighter group (Grigorchuk-Zuk)

Example 2: Approximation of laplacians on fractals. E.g. $\text{Julia}(z^2-1)$ (Teplyaev-Rogers)
II. Random weak limit (RWL) of finite graphs. (Benjamini-Schramm).

Let $G$ be a finite connected graph. Consider it as rooted by choosing a root uniformly at random. This defines a probability measure $\lambda_G$ on the space $G_*$ of rooted graphs. Namely, for $[(H,x)] \in G_*$, $\lambda_G([(H,x)])$ is equal to the proportion of vertices $y$ of $G$ such that the rooted graph $(G,y)$ is rooted isomorphic to $(H,x)$.

Suppose now that $(G_n)$ are finite connected graphs of bounded degree with and that $\mu$ is a probability measure on $G_*$. We say that the RWL of $(G_n)$ is $\mu$ if $\lambda_n$ converges to $\mu$ weakly in $G_*$. 
Note that if $G$ is transitive, then $\lambda_G$ is a $\delta$-measure, concentrated on $[(G,x)]$ for any $x \in \text{Vert}(G)$. If the random weak limit $\mu$ is concentrated on one transitive graph (rooted at any vertex), then we say that the random weak limit of $(G_n)$ is $G$.

**Example.** RWL of discrete circles on $2^n$ vertices is $\mathbb{Z}$.

**Question (Aldous-Lyons):** what measures on $G_*$ arise as random weak limits of sequences of finite graphs?

In particular, can a Cayley graph of an arbitrary finitely generated group be approximated in this sense by finite graphs?

Is every finitely generated group sofic?
Random weak limits of Schreier graphs

Let \((\Gamma_n)\) be as before a sequence of Schreier graphs of a self-similar action of \(G<\text{Aut}(T)\). RWL of \((\Gamma_n)\) is then a probability measure concentrated on rooted isomorphism classes of orbital Schreier graphs \(\{\Gamma_\xi | \xi \in \partial T\}\).

**Question:** under which conditions on (the action of) \(G\) the RWL is continuous (=non-atomic)?

**Remark.** RWL continuous \(\iff\) existence of a continuous ergodic probability measure on the space \(\text{Sub}(G)\) of all subgroups of \(G\), invariant under the action of \(G\) on \(\text{Sub}(G)\) by conjugation.
Problem: Understand isomorphism classes of orbital Schreier graphs \( \{\Gamma_\xi \mid \xi \in \partial T\} \) for the action of \( G \) on the boundary \( \partial T \).

Example 1. Adding machine. \( \Gamma_n \) is a circle on \( 2^n \) vertices, \( \Gamma_\xi \) is the discrete line \( \mathbb{Z} \) for all \( \xi \).

More generally, if the action of \( G \) on \( \partial T \) is ess. free, then almost all Schreier graphs \( \Gamma_\xi \) are isomorphic to the Cayley graph of \( G \) w.r.t. \( S \).

Example 2. Grigorchuk’s group. \( \Gamma_\xi, \forall \xi \neq w111... \)

... \[\cdots\] ...

But all Schreier graphs labelled by \( S \) are different.
Example 3. Basilica group.

Uncountably many non-isomorphic orbital Schreier graphs. Almost every isomorphism class contains uncountably many different orbits.

(D’Angeli, Donno Matter, Nagnibeda, JMD’10)
Almost every orbital Schreier graph has one end. There is also an uncountable family of non-isomorphic orbital Schreier graphs with two ends and one graph with 4 ends.

A finite part of $\Gamma_{odd}$. 
The infinite Schreier graph $(\Gamma, 0^\omega)$. 
Example 4. Hanoi towers group $H^{(3)}$

Uncountably many isomorphism classes of orbital Schreier graphs. Each isomorphism class contains only finitely many orbits.

The graph $P$ is obtained by deleting all even numbers from the Pascal triangle and joining each odd number to the closest odd neighbors. It also arises naturally in the Hanoi Towers game.
Questions. For a contracting self-similar group acting on a regular rooted tree,

1) How to characterize groups (exceptions?) like in Example 2? – Relation to topological and metric properties of the limit space?

Note: in examples 1 and 2 the limit spaces are the circle and the interval, in examples 3 and 4 the limit spaces are fractal.

2) How to distinguish between situations as in Examples 3 and 4? (Ex. 4 also includes Gupta-Fabrikowski group, Thompson’s group F (Savchuk), etc.)
The action of $G$ on $\partial T$ is ergodic w.r.t. the uniform measure $\lambda$ on $\partial T$, and therefore the “typical” number of ends for a given self-similar action is well-defined. We study both the “typical number” $E(G)$ and the number of ends $E(\xi)$ of any individual Schreier graph $\Gamma_\xi$, $\xi \in \partial T$.

Assumption. We assume that $G$ is generated by a bounded automaton, i.e., that the number of vertices $v$ on the $n$-th level of the tree with nontrivial activity $s|_v \neq e_G$ remains bounded for every state $s \in S$ of the automaton. (Recall $G=<S>$).
“Typical” number of ends $E(G)$

Obs.: In a RWL of a sequence of finite graphs, the “typical” w.r.t. RWL number of ends is well-defined and is a.s. equal to $0, 1, 2$, or $\infty$. Moreover, if the limit graphs are all amenable (NB: all bounded automata groups are amenable), then the “typical” number is $0, 1$, or $2$.

Therefore, for a given bounded automaton group $G$, the “typical” number $E(G)$ of ends in an orbital Schreier graph is $0$, $1$ or $2$.

Theorem. Characterization of finite bounded automata with $E(G)=2$. 
Def. A sequence $x_2x_1 \in X^{-\omega}$ is postcritical if it can be read on an infinite path in the Moore diagram of the nucleus of $G$, beginning in a non-trivial state and going in the sense opposite to the arrows. Let $P$ denote the set of postcritical sequences. The automaton has bounded activity if and only if $|P|<\infty$.

Theorem. 1) There are only fin. many orbital Schreier graphs with more than 2 ends.

2) $|P|=1 \implies G$ is finite $\implies E(G)=0$.

3) $|P| \geq 3 \implies E(G)=1$. 
4) \( E(G) = 2 \) iff \( |P| = 2 \) and, (up to passing to a power of the alphabet) the nucleus of the group is one of the following:

a. is the adding machine

b. consists of automorphisms of type II and III (+ some condition on the orbits)

c. consists of automorphisms of type III and IV (+ some condition on the orbits)
Relation with the limit space

**Theorem.** a) characterizes contracting groups with the limit space a circle. b) characterizes contracting groups with limit space an interval.

**Rem.** Such groups are also characterized algebraically: (Nekrashevych-Sunic)

**Theorem.** The number of ends in a “typical” orbital Schreier graph is equal to the number of connected components in the complement of a “typical” point in the limit space.

(Such quantities were considered for Julia sets of quadratic polynomials, by Zdunik etc.)
Other results. For every group generated by a bounded automaton, we construct
- a finite automaton (states = partitions of the postcritical set $P$) that, reading an infinite sequence $\xi \in \partial T$, returns the number of ends in the orbital Schreier graph $\Gamma_\xi$.
- a finite automaton that, given a cut point in the limit space, returns its degree.
- provide a description of graphs with more than two ends (set of measure zero, critical points).

Such information is useful in rigidity results – as e.g. in: Bartholdi-Nekrahnsevych.