Group acting on Λ trees

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Montreal, 2010

This talk is based on joint results with A. Myasnikov and D. Serbin.

The starting point

Theorem . (Babai, Serre) A group G is free if and only if it acts freely on a tree.

Free action = no inversion of edges and stabilizers of vertices are trivial.

Ordered abelian groups

$\Lambda =$ an ordered abelian group (any $a, b \in \Lambda$ are comparable and for any $c \in \Lambda$: $a \leq b \Rightarrow a + c \leq b + c$).

Examples:

Archimedean case:

$\Lambda = \mathbb{R}$, $\Lambda = \mathbb{Z}$ with the usual order.

Non-Archimedean case:

$\Lambda = \mathbb{Z}^2$ with the right lexicographic order:

$$(a, b) < (c, d) \iff b < d \text{ or } b = d \text{ and } a < c.$$

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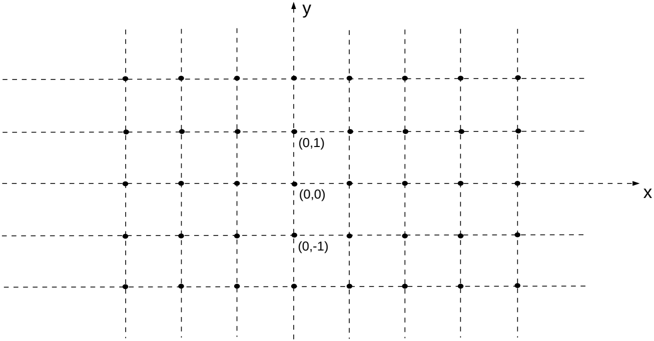
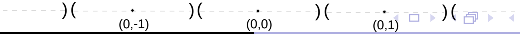
$\mathbb{Z}^2$ with the right-lex ordering



One-dimensional picture



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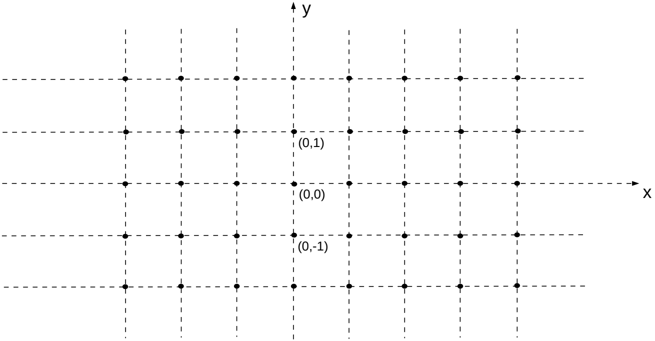

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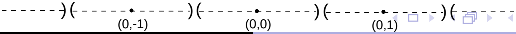

Λ-trees

Morgan and Shalen (1985) defined Λ-trees:
A Λ-tree is a metric space (X, p) (where $p : X \times X \to \Lambda$) which satisfies the following properties:

1) (X, p) is geodesic,
2) if two segments of (X, p) intersect in a single point, which is an endpoint of both, then their union is a segment,
3) the intersection of two segments with a common endpoint is also a segment.

Alperin and Bass (1987) developed the theory of Λ-trees and stated the fundamental research goals:

Find the group theoretic information carried by an action on a Λ-tree.

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Generalize Bass-Serre theory (for actions on $\mathbb{Z}$-trees) to actions on arbitrary Λ-trees.

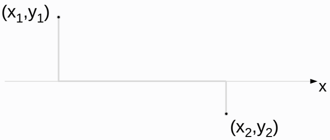
Examples for $\Lambda = \mathbb{R}$

$X = \mathbb{R}$ with usual metric.

A geometric realization of a simplicial tree.

$X = \mathbb{R}^2$ with metric d defined by

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1| + |y_2| + |x_1 - x_2| & \text{if } x_1 \neq x_2 \\ |y_1 - y_2| & \text{if } x_1 = x_2 \end{cases}$$



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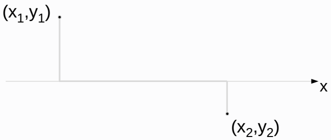

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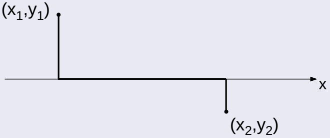

Finitely generated $\mathbb{R}$-free groups

Rips' Theorem [Rips, 1991 - not published]

A f.g. group acts freely on $\mathbb{R}$-tree if and only if it is a free product of surface groups (except for the non-orientable surfaces of genus 1,2, 3) and free abelian groups of finite rank.

Gaboriau, Levitt, Paulin (1994) gave a complete proof of Rips' Theorem.

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Properties

Some properties of groups acting freely on Λ-trees (Λ-free groups)

1. The class of Λ-free groups is closed under taking subgroups and free products.
2. Λ-free groups are torsion-free.
3. Λ-free groups have the CSA-property (maximal abelian subgroups are malnormal).
4. Commutativity is a transitive relation on the set of non-trivial elements.
5. Any two-generator subgroup of a Λ-free group is either free or free abelian.

The Fundamental Problem

The following is a principal step in the Alperin-Bass' program:

Open Problem [Rips, Bass]

Describe finitely generated groups acting freely on Λ-trees.

Here "describe" means "describe in the standard group-theoretic terms".

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Theorem (H.Bass, 1991)

A finitely generated $(\Lambda \oplus \mathbb{Z})$-free group is the fundamental group of a finite graph of groups with properties:

- vertex groups are Λ-free,
- edge groups are maximal abelian (in the vertex groups),
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Actions on $\mathbb{R}^n$-trees

Theorem [Guirardel, 2003]

A f.g. freely indecomposable $\mathbb{R}^n$-free group is isomorphic to the fundamental group of a finite graph of groups, where each vertex group is f.g. $\mathbb{R}^{n-1}$-free, and each edge group is cyclic.

However, the converse is not true.

Corollary A f.g. $\mathbb{R}^n$-free group is hyperbolic relative to abelian subgroups.

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$\mathbb{Z}^n$-free groups

Theorem [Kharlampovich, Miasnikov, Remeslennikov, 96]

Finitely generated fully residually free groups are $\mathbb{Z}^n$-free.

Theorem [Martino and Rourke, 2005]

Let G_1 and G_2 be $\mathbb{Z}^n$-free groups. Then the amalgamated product $G_1 *_C G_2$ is $\mathbb{Z}^m$-free for some $m \in \mathbb{N}$, provided C is cyclic and maximal abelian in both factors.

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Examples of $\mathbb{Z}^n$-free groups:

$\mathbb{R}$-free groups,

$\langle\, x_1, x_2, x_3 \mid x_1^2\ x_2^2\ x_3^2 = 1 \,\rangle$ is $\mathbb{Z}^2$-free (but is neither $\mathbb{R}$-free, nor fully residually free).

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The Main Conjecture.

Conjecture

Every finitely generated Λ-free group is $\mathbb{Z}^n$-free.

How probable it is?

Theorem [Kharlampovich, Myasnikov, Serbin]

Every finitely generated $\mathbb{R}$-free group is $\mathbb{Z}^2$-free for some $m \geq n$.

All known finitely generated Λ-free groups are $\mathbb{Z}^m$-free.

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From actions to length functions

Let G be a group acting on a Λ-tree (X, d). Fix a point $x_0 \in X$ and consider a function $l : G \longrightarrow \Lambda$ defined by

$$l(g) = d(x_0\ ,\ gx_0)$$

l is called a based length function on G with respect to x_0, or a Lyndon length function.

l is free if the underlying action is free.

Example. In a free group F, the function $f \to |f|$ is a free $\mathbb{Z}$-valued (Lyndon) length function.

Length functions

Length functions were introduced by **Lyndon (1963)**.

Let G be a group. A function $l : G \to \Lambda$ is called a length function on G if

(L1) $\forall\, g \in G: \ l(g) \geqslant 0$ and $l(1) = 0$,

(L2) $\forall\, g \in G: \ l(g) = l(g^{-1})$,

(L3) the triple $\{c(g,f), c(g,h), c(f,h)\}$ is isosceles for all $g, f, h \in G$, where $c(f,g)$ is the Gromov's product:

$$c(g,f) = \frac{1}{2}(l(g) + l(f) - l(g^{-1}f)).$$

$\{a, b, c\}$ is isosceles = at least two of a, b, c are equal, and not greater than the third.

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Examples of length functions

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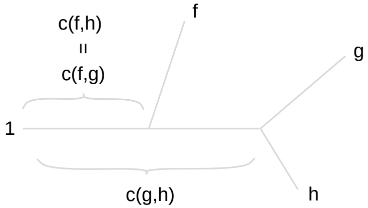

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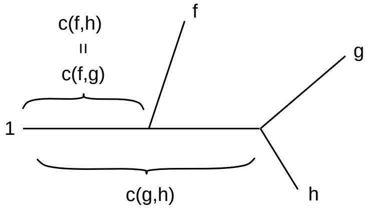

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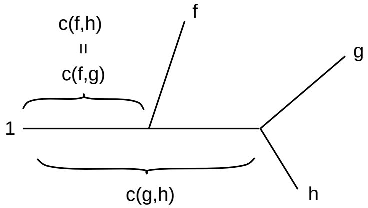

Free length functions

A length function $l : G \to \Lambda$ is free if $l(g^2) > l(g)$ for every non-trivial $g \in G$.

Chiswell's Theorem

Theorem [Chiswell]

Let $L : G \to \Lambda$ be a Lyndon length function on a group G. Then there exists a Λ-tree (X, d), $x \in X$, and an isometric action of G on X such that $L(g) = d(x, gx)$ for all $g \in G$.

Notice that $L(g) = d(x, gx)$ is free iff the action of G on X is free.

This gives another approach to free actions on Λ-trees.

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Infinite words

Let Λ be a discretely ordered abelian group with a minimal positive element 1_Λ and $X = \{x_i \mid i \in I\}$ be a set.

An **Λ-word** is a function

$$w : [1_\Lambda, \alpha] \to X^\pm, \quad \alpha \in \Lambda.$$

$|w| = \alpha$ is called the length of w.

w is **reduced** $\Longleftrightarrow$ no subwords xx^{-1}, $x^{-1}x$ $(x \in X)$.

$R(\Lambda, X) =$ the set of all reduced Λ-words.

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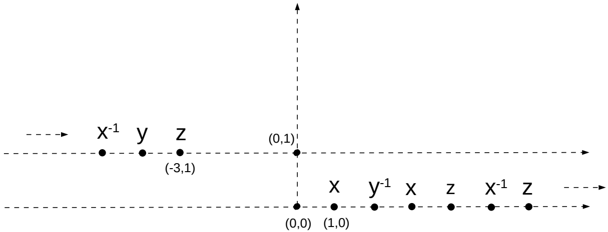
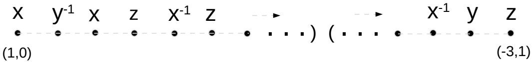
Example.

Let $X = \{x, y, z\}$, $\Lambda = \mathbb{Z}^2$



In "linear" notation



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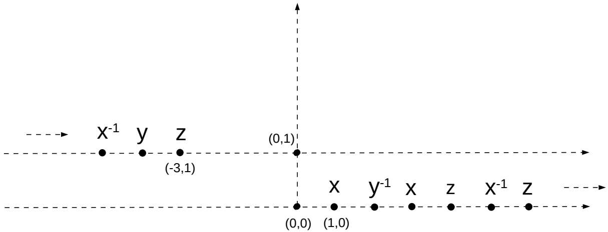

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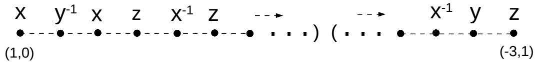
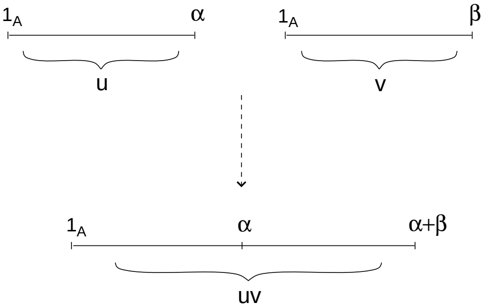
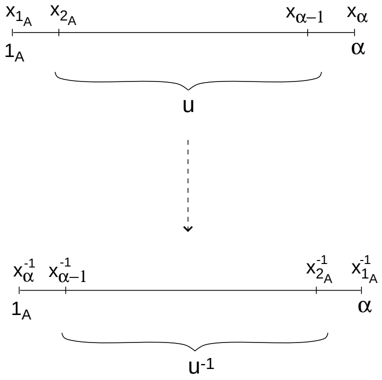


Concatenation of Λ-words:

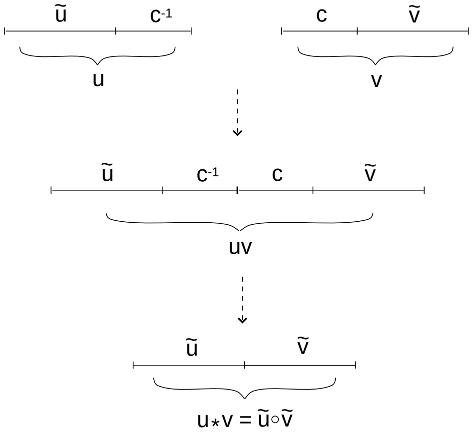


We write $u \circ v$ instead of uv in the case when uv is reduced.

Inversion of Λ-words:



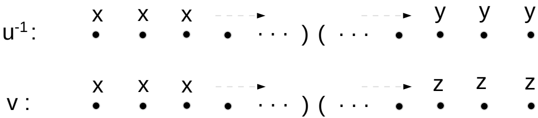
Multiplication of Λ-words:



The partial group $R(\Lambda, X)$

The multiplication on $R(\Lambda, X)$ is partial, it is not everywhere defined!

Example. $u, v \in R(\mathbb{Z}^2, X)$



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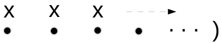

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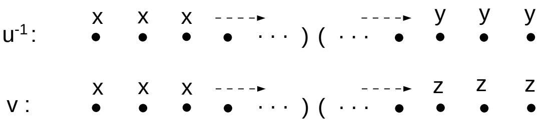

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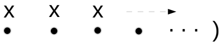

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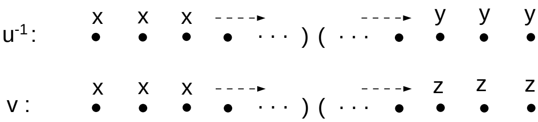

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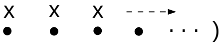

Cyclic decompositions

$v \in R(\Lambda, X)$ is **cyclically reduced** if $v(1_A)^{-1} \neq v(|v|)$.

$v \in R(\Lambda, X)$ admits a **cyclic decomposition** if

$$v = c^{-1} \circ u \circ c,$$

where $c, u \in R(A, \Lambda)$ and u is cyclically reduced.

Denote by $CDR(A, \Lambda)$ the set of all words from $R(\Lambda, X)$ which admit a cyclic decomposition.

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From Non-Archimedean words - to length functions

Theorem [Myasnikov-Remeslennikov-Serbin, 2003]

Let Λ be a discretely ordered abelian group and X a set. If G is a subgroup of $CDR(\Lambda, X)$ then the function $L_G : G \to \Lambda$, defined by $L_G(g) = |g|$, is a free Lyndon length function.

Corollary.

To show that a group G acts on a Λ-tree - embed G into $CDR(\Lambda, X)$.

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From Length functions - to Non-Archimedean words

Theorem [Chiswell], 2004

Let Λ be a discretely ordered abelian group. If $L : G \to \Lambda$ is a free Lyndon length function on a group G then there exists an embedding $\phi : G \to CDR(\Lambda, X)$ such that $|\phi(g)| = L(g)$ for every $g \in G$.

Corollary. Let Λ be an arbitrary ordered abelian group. If $L : G \to \Lambda$ is a free Lyndon length function on a group G then there exists a length preserving embedding $\phi : G \to CDR(\Lambda', X)$, where $\Lambda' = \Lambda \oplus \mathbb{Z}$ with the lex order.

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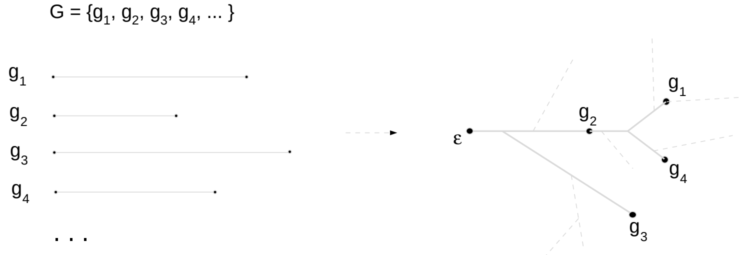
From Non-Archimedean words - to free actions

Infinite words $\Longrightarrow$ Length functions $\Longrightarrow$ Free actions

Shortcut

If $G \hookrightarrow CDR(\Lambda, X)$ then G acts by isometries on the canonical Λ-tree $\Gamma(G)$ labeled by letters from $X^{\pm}$.



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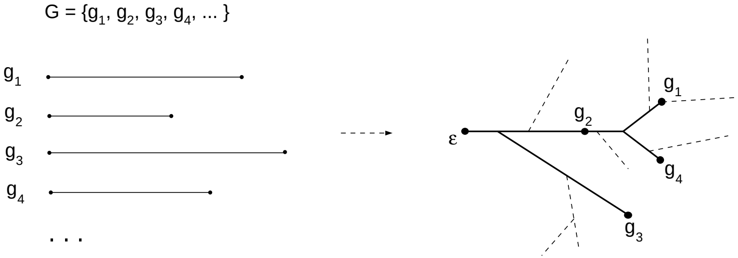

Regular free actions

A length function $l : G \to A$ is called *regular* if it satisfies the *regularity* axiom:

(L6) $\forall\, g, f \in G,\ \exists\, u, g_1, f_1 \in G$:

$$g = u \circ g_1 \ \&\ f = u \circ f_1 \ \&\ l(u) = c(g, f).$$

Complete subgroups

Let $G \leq CDR(\Lambda, X)$ be a group of infinite words.

Complete subgroups

$G \leq CDR(\Lambda, X)$ is complete if G contains the common initial segment $c(g, h)$ for every pair of elements $g, h \in G$.

Regular length functions

A Lyndon length function $L : G \to \Lambda$ is regular if there exists a length preserving embedding $G \to CDR(\Lambda, X)$ onto a complete subgroup.

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Complete subgroups

Example. Let $F(x, y)$ be a free group and $H = \langle x^2 y^2,\ xy \rangle$ be its subgroup.

F has natural free $\mathbb{Z}$-valued length function $l_F : f \to |f|$. Hence, l_F induces a length function l_H on H.

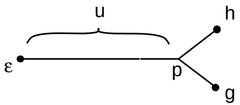
l_F is regular, but l_H is not

Take $g = xy^{-1}x^{-2},\ h = xy^{-1}x^{-1}y$ in F. Then

$$g, h \in H, \quad \text{but} \quad com(g, h) = xy^{-1}x^{-1} \notin H.$$

Branch points and completeness

A vertex $p \in \Gamma(G)$ is a branch point if it is the terminal endpoint of the common initial segment $u = com(g, h)$ of $g, h \in G$.



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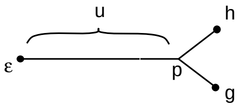

Completeness $\Longrightarrow$ all branch points are in one G-orbit of Γ

Finitely presented complete Λ-free groups.

Theorem [Kharlampovich, Myasnikov, Serbin, 2008]

If G is f.p. and has a regular free length function in Λ, then G can be represented as a union of a finite series of groups

$$G_1 < G_2 < \cdots < G_n = G,$$

where

1. G_1 is a free group,
2. G_{i+1} is obtained from G_i by finitely many HNN-extensions in which associated subgroups are length-isomorphic maximal abelian.

The General Case

Chiswell, 2010

Every finitely generated Λ-free group is a subgroup of a complete Λ-free group.

Kharlampovich, Miasnikov, Serbin, 2010

Every finitely generated $\mathbb{Z}^n$-free group is a subgroup of a f.g. complete $\mathbb{Z}^n$-free group.

Conjecture

Every finitely generated complete Λ-free group is finitely presented.

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Elimination process

Elimination Process (EP) is a dynamical (rewriting) process of a certain type that transforms formal systems of equations in groups or semigroups (or band complexes, or foliated 2-complexes, or partial isometries of multi-intervals) .

Makanin (1982): Initial version of EP.

Makanin's EP gives a decision algorithm to verify consistency of a given system of equations - decidability of the Diophantine problem over free groups.

Makanin introduced the fundamental notions: generalized equations, elementary and entire transformations, notion of complexity.

Razborov's process

Razborov (1987): developed EP much further.
Razborov's EP produces **all solutions** of a given system in F.
The *coordinate group* of $S = 1$: $F_{R(S)} = F(A \cup X)/Rad(S)$

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The *coordinate group* of $S = 1$: $F_{R(S)} = F(A \cup X)/Rad(S)$

$$
\begin{tikzcd}
 & & F_{R(S)} \arrow[dl] \arrow[dr] \arrow[drrr] & & & \\
 & F_{R(\Omega_{v_1})} \arrow[loop above, "\sigma_1"] \arrow[dl] \arrow[dr] & & F_{R(\Omega_{v_2})} & \cdots & F_{R(\Omega_{v_n})} \\
F_{R(\Omega_{v_{21}})} & \cdots & F_{R(\Omega_{v_{2m}})} \arrow[loop above, "\sigma_2"] \arrow[dl] \arrow[dr] & & & \\
 & {} & \cdots & {} & & \\
 & \cdots & F_{R(\Omega_{v_k})} \arrow[d] & & & \\
 & & F(A) * F(T) & & &
\end{tikzcd}
$$

Kharlampovich - Myasnikov (1998):

Refined Razborov's process.
Effective description of solutions of equations in free (and fully residually free) groups in terms of very particular **triangular systems** of equations.
Resembles the classical elimination theory for polynomials.

Elimination process and JSJ

A **splitting** of G is a representation of G as the fundamental groups of a graph of groups.
A splitting is **cyclic (abelian)** if all the edge groups are cyclic (abelian).

Elementary splittings:

$G = A *_C B, \quad G = A*_C = \langle A, t \mid t^{-1}Ct = C' \rangle,$

Free splittings:

$G = A * B$

Grushko's decompositions

All free splittings of G are encoded in **Grushko's decompositions**.
A free decomposition

$$G = G_1 * \ldots * G_k * F_r$$

is a Grushko's decomposition of G if $G_1, \ldots, G_k$ are freely indecomposable non-cyclic groups and F_r is a free group of rank r.
Grushko's decompositions are **essentially unique**.

JSJ decompositions

All cyclic (abelian) splittings of G are encoded in **JSJ decompositions** of G.

JSJ decompositions are universal decompositions with vertices of the following types: **QH-vertices, abelian, rigid**.

JSJ decompositions are **essentially unique**.

Infinite branches and JSJ

Motto: JSJ is an algebraic counterpart of EP.
infinite branches of EP $\Longleftrightarrow$ **abelian splittings** of the coordinate groups of the systems.

Moreover, the automorphisms associated with infinite branches of the process are precisely the canonical automorphisms of the JSJ decomposition associated with the splittings.

Effectiveness of Grushko's and JSJ decompositions

Theorem [Touikan, 2010] *There is an algorithm which for every finitely presented group with solvable word problem finds its Grushko's decomposition (by giving finite generating sets of the factors).*

Theorem [KM] *There exists an algorithm to obtain a cyclic [abelian] JSJ decomposition of a freely indecomposable fully residually free group. The algorithm constructs a presentation of this group as the fundamental group of a JSJ graph of groups.*

Hint of the proof

Let $G = F_{R(S)}$ be a finitely generated fully residually free group.
Solutions of $S(X, A) = 1$ in $F \iff$ homomorphisms $\phi : G \to F$.

Composition of ϕ with $\sigma \in Aut(G) \Longrightarrow$ a new solution of $S(X, A) = 1$ in F.

Canonical automorphisms associated with a JSJ decomposition of $G \iff$ solutions of the system $S(X, A) = 1$ of a particular type.

One can recognize these solutions in EP as infinite branches.
Infinite branches $\Longleftrightarrow$ splittings of G.
Bass-Serre Theory + length functions techniques $\Longrightarrow$ JSJ decompositions of G.

Isomorphism problem

Bumagin, Kharlampovich, Myasnikov *The isomorphism problem is decidable in the class of all finitely generated fully residually free groups.*

Dahmani and Groves The isomorphism problem is decidable in the class of torsion-free groups which are hyperbolic relative to abelian subgroups.

The case of $\mathbb{Z}^n$-free groups.

Theorem. *Let G be a finitely generated $\mathbb{Z}^n$-free group. Then:*

- *(Nikolaev, Serbin, 2010) There is an algorithm to solve the membership problem,*
- *There is an algorithm to obtain a cyclic [abelian] JSJ decomposition of a freely indecomposable G.*
- *The Isomorphism Problem is decidable in the class of finitely generated $\mathbb{Z}^n$-free groups.*

To prove 2,3 one needs an elimination process in this class **(KM)**. It also follows from **Dahmani and Groves** since these groups are hyperbolic relative to abelian subgroups.

Elimination processes and free actions

Infinite branches of an elimination process correspond precisely to the standard types of free actions:
linear case $\Longleftrightarrow$ **thin (or Levitt)** type
the quadratic case $\Longleftrightarrow$ **surface type (or interval exchange)**,
periodic structures $\Longleftrightarrow$ **toral (or axial)** type.

Bestvina-Feighn's elimination process

A powerful variation of the Makanin-Razborov's process for $\mathbb{R}$-actions.

Can be viewed as an asymptotic (limit) version of MR process.

Much simpler in applications but not algorithmic.

KM elimination process for $\mathbb{Z}^n$ actions

To solve equations in fully residually free groups we designed a variation of the elimination process for $\mathbb{Z}^n$ actions.

It **effectively** describes solution sets of finite systems of equations in $\mathbb{Z}^n$-groups in terms of **Triangular quasi-quadratic systems** (as in the case of fully residually free groups).

Non-standard version of Rip's machine

Kh., Myasnikov, and Serbin designed an elimination process for arbitrary non-Archimedean actions, i.e, free actions on Λ-trees.

This can be viewed as a **non-Archimedean (non-standard)** discrete, effective version of the original MR process.

Sketch of the proof of the theorem about Λ-free f.p. groups

Let G have a regular free length function in Λ.

Fix an embedding of G into $CDR(\Lambda, X)$ and construct a cancellation tree for each relation of G.

Sketch of the proof

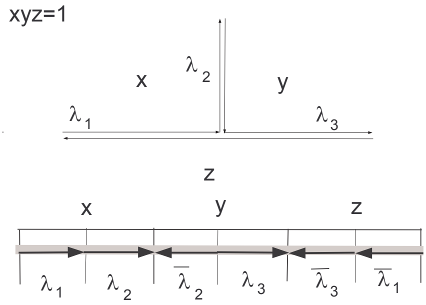

Figure: From the cancellation tree for the relation $xyz = 1$ to the generalized equation $(x = \lambda_1 \circ \lambda_2,\ y = \lambda_2^{-1} \circ \lambda_3,\ z = \lambda_3^{-1} \circ \lambda_1^{-1})$.

Sketch of the proof

Infinite branches of an elimination process correspond to abelian splittings of G:

linear case $\Longleftrightarrow$ **splitting as a free product.**

the quadratic case $\Longleftrightarrow$ **QH-subgroup**,

periodic structures $\Longleftrightarrow$ **abelian** vertex group or splitting as an HNN with abelian edge group.

After obtaining a splitting we apply EP to the vertex groups. We build the Delzant-Potyagailo hierarchy.

Sketch of the proof

A family $\mathcal{C}$ of subgroups of a torsion-free group G is called *elementary* if

(a) $\mathcal{C}$ is closed under taking subgroups and conjugation,
(b) every $C \in \mathcal{C}$ is contained in a maximal subgroup $\overline{C} \in \mathcal{C}$,
(c) every $C \in \mathcal{C}$ is small (does not contain F_2 as a subgroup),
(d) all maximal subgroups from $\mathcal{C}$ are malnormal.

G admits a *hierarchy* over $\mathcal{C}$ if the process of decomposing G into an amalgamated product or an HNN-extension over a subgroup from $\mathcal{C}$, then decomposing factors of G into amalgamated products and/or HNN-extensions over a subgroup from $\mathcal{C}$ etc. eventually stops.

Theorem (Delzant - Potyagailo (2001)). If G is a finitely presented group without 2-torsion and $\mathcal{C}$ is a family of elementary subgroups of G then G admits a hierarchy over $\mathcal{C}$.

Corollary. If G is a f.p. Λ-free group then G admits a hierarchy

Hyperbolic length functions Let G be a group, Λ an ordereg abelian group. A function $l : G \to \Lambda$ is called a δ-hyperbolic length function on G if

(L1) $\forall\ g \in G :\ l(g) \geqslant 0$ and $l(1) = 0$,

(L2) $\forall\ g \in G :\ l(g) = l(g^{-1})$,

(L3) $\forall\ g, h \in G :\ l(gh) \leq l(g) + l(h)$,

(L4) $\forall\ f, g, h \in G :\ c(f, g) \geq min\{c(f, h), c(g, h)\} - \delta$, where $c(f, g)$ is the Gromov's product:

$$c(g, f) = \frac{1}{2}(l(g) + l(f) - l(g^{-1}f)).$$

A δ-hyperbolic length function is called complete if $\forall g \in G$, and $\alpha \leq l(g)$ there is $u \in G$ such that $g = u \circ g_1$, where $l(u) = \alpha$.

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Free hyperbolic length functions A δ -hyperbolic length function is called free (δ-free) if
$\forall g \in G : \ g \neq 1 \rightarrow l(g^2) > l(g)$ (*resp.*, $l(g^2) > l(g) - c(\delta)$).

Problem

Find the structure of f.g. groups with δ-hyperbolic, δ-regular, δ-free length function, in $\mathbb{Z}^n$, where $l(\delta)$ is in the smallest component of $\mathbb{Z}^n$.

A.P. Grecianu (McGill) obtained first results in this direction.