The topology and geometry of Outer space

Mladen Bestvina

$Out(F_n)$

$F_n = < a_1, a_2, \cdots, a_n >$ is the free group of rank n.

$$Out(F_n) = Aut(F_n)/Inn(F_n)$$

- contains $MCG(S)$ for punctured surfaces S
- maps to $GL_n(\mathbb{Z})$

The study of mapping class groups and arithmetic groups is an inspiration in the study of $Out(F_n)$.

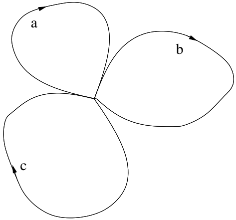
Theorem (Nielsen, 1924)

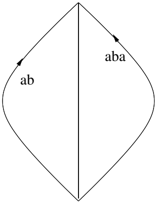
$Aut(F_n)$ (and $Out(F_n)$) are finitely presented. A generating set consists of the automorphisms σ: $a_1 \mapsto a_1a_2$, $a_i \mapsto a_i$ for $i > 1$ plus the signed permutations of the a_i's.

Outer space

Definition

- graph: finite 1-dimensional cell complex Γ, all vertices have valence ≥ 3.
- rose $R = R_n$: wedge of n circles.





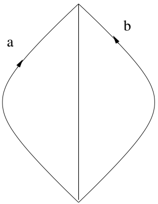
- marking: homotopy equivalence $g : \Gamma \to R$.
- metric on Γ: assignment of positive lengths to the edges of Γ so that the sum is 1.

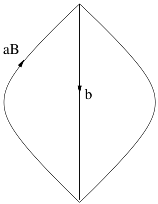
Outer space

Definition (Culler-Vogtmann, 1986)

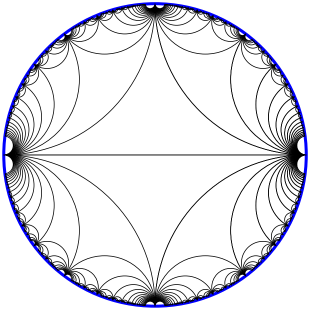
Outer space X_n is the space of equivalence classes of marked metric graphs (g, Γ) where $(g, \Gamma) \sim (g', \Gamma')$ if there is an isometry $\phi : \Gamma \to \Gamma'$ so that $g'\phi \simeq g$.

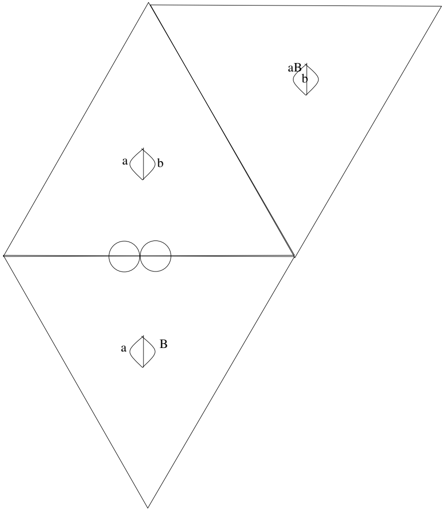
$$\begin{tikzcd} \Gamma \arrow[dr, "g"] \arrow[dd, "\phi"'] & \\ & R \\ \Gamma' \arrow[ur, "g'"'] & \end{tikzcd}$$

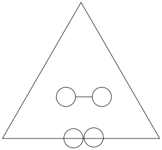




Outer space in rank 2







Triangles have to be added to edges along the base.

Outer space

Topology (3 approaches, all equivalent):

- simplicial, with respect to the obvious decomposition into "simplices with missing faces".
- (g, Γ) is close to (g', Γ') if there is a $(1 + \epsilon)$-Lipschitz map $f : \Gamma \to \Gamma'$ with $g'f \simeq g$.
- via length functions: if α is a conjugacy class in F_n let $\ell_{(g,\Gamma)}(\alpha)$ be the length in Γ of the unique immersed curve a such that $g(a)$ represents α. Then, for S =set of conjugacy classes

$$X_n \to [0, \infty)^S$$

$$(g, \Gamma) \mapsto (\alpha \mapsto \ell_{(g,\Gamma)}(\alpha))$$

  is injective – take the induced topology.

Theorem (Culler-Vogtmann, 1986)

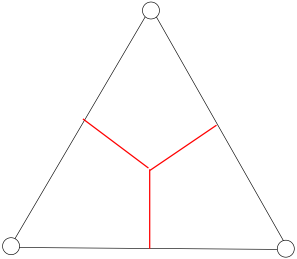
X_n is contractible.

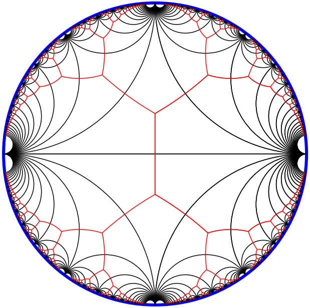
Action

If $\phi \in Out(F_n)$ let $f : R \to R$ be a h.e. with $\pi_1(f) = \phi$ and define

$$\phi(g, \Gamma) = (fg, \Gamma) \qquad\qquad \Gamma \xrightarrow{g} R_n \xrightarrow{f} R_n$$

- action is simplicial,
- point stabilizers are finite.
- there are finitely many orbits of simplices (but the quotient is not compact).
- the action is cocompact on the spine $SX_n \subset X_n$.





Topological properties

Finiteness properties:

- Virtually finite $K(G, 1)$ (Culler-Vogtmann 1986).
- $vcd(Out(F_n)) = 2n - 3$ ($n \geq 2$) (Culler-Vogtmann 1986).
- every finite subgroup fixes a point of X_n.

Other properties:

- every solvable subgroup is finitely generated and virtually abelian (Alibegović 2002)
- Tits alternative: every subgroup $H \subset Out(F_n)$ either contains a free group or is virtually abelian (B-Feighn-Handel, 2000, 2005)
- Bieri-Eckmann duality (B-Feighn 2000)

$$H^i(G; M) \cong H_{d-i}(G; M \otimes D)$$

- Homological stability (Hatcher-Vogtmann 2004)

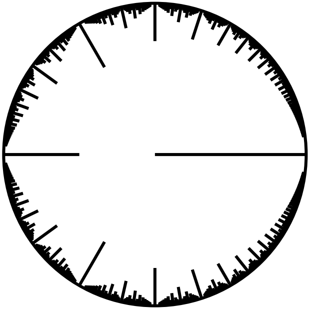
$$H_i(Aut(F_n)) \cong H_i(Aut(F_{n+1})) \text{ for } n >> i$$

- Computation of stable homology (Galatius, to appear)

Dynamical properties

- X_n can be equivariantly compactified to $\overline{X_n}$ (Culler-Morgan, 1987), analogous to Thurston's compactification of Teichmüller space via projective measured laminations.
- $$X_n \subset [0, \infty)^S \to P[0, \infty)^S$$

  is injective; take the closure.



- A point of X_n can be viewed as a free simplicial F_n-tree; a point in $\partial X_n = \overline{X_n} \setminus X_n$ is an F_n-tree (not necessarily free nor simplicial).

Dynamical properties

- Points in ∂X_n can be studied using the Rips machine.
- Guirardel (2000): action on ∂X_n does not have dense orbits. He also conjecturally identified the minimal closed invariant set.
- North-South dynamics for fully irreducible elements (Levitt-Lustig, 2003)

Dictionary 1

$SL_2(\mathbb{Z})$	$SL_n(\mathbb{Z})$	$MCG(S)$	$Out(F_n)$
Trace	Jordan normal form	Nielsen-Thurston theory	train-tracks
$\mathbb{H}^2$	symmetric space	Teichmüller space	Outer space
hyperbolic (Anosov) element	semi-simple (diagonalizable)	pseudo-Anosov mapping class	fully irreducible automorphism
shear	parabolic	Dehn twist	polynomially growing automorphism

Dictionary 2

$MCG(S)$	$Out(F_n)$
simple closed curve	primitive conjugacy class
incompressible subsurface	free factor splitting of F_n
measured lamination	$\mathbb{R}$-tree
Thurston's boundary	Culler-Morgan's boundary
attracting lamination for a pseudo-Anosov	attracting tree for a fully irreducible automorphism
measured geodesic current	measured geodesic current
intersection number between measured laminations	length of a current in an $\mathbb{R}$-tree
curve complex	free factor complex splitting complex

Lipschitz metric on Outer space

Three metrics on Teichmüller space:

- Teichmüller metric,
- Weil-Petersson metric,
- Thurston's Lipschitz metric.

Only the Lipschitz metric has an analog.
If $(g, \Gamma), (g', \Gamma') \in X_n$ consider maps $f : \Gamma \to \Gamma'$ so that $g'f \simeq g$ (such f is the difference of markings).

$$\begin{tikzcd} \Gamma \arrow[d, "f"'] \arrow[dr, "g"] & \\ \Gamma' \arrow[r, "g'"'] & R \end{tikzcd}$$

Consider only f's that are linear on edges.
Arzela-Ascoli $\Rightarrow \exists f$ that minimizes the largest slope, call it $\sigma(\Gamma, \Gamma')$.

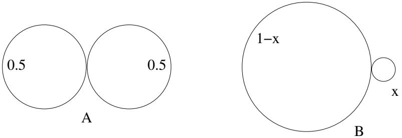
Lipschitz metric on Outer space

Definition

$d(\Gamma, \Gamma') = \log \sigma(\Gamma, \Gamma')$

- $d(\Gamma, \Gamma'') \leq d(\Gamma, \Gamma') + d(\Gamma', \Gamma'')$,
- $d(\Gamma, \Gamma') = 0 \iff \Gamma = \Gamma'$.
- in general, $d(\Gamma, \Gamma') \neq d(\Gamma', \Gamma)$.

Example



$$d(A, B) = \log \frac{1-x}{0.5} \to \log 2$$

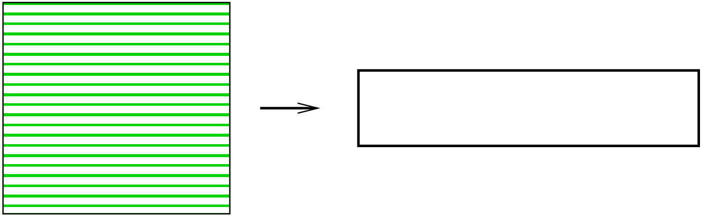
$$d(B, A) = \log \frac{0.5}{x} \to \infty$$

But [Handel-Mosher] The restriction of d to the spine is quasi-symmetric, i.e. $d(\Gamma, \Gamma')/d(\Gamma', \Gamma)$ is uniformly bounded.

Lipschitz metric on Outer space

Theorem (Thurston)

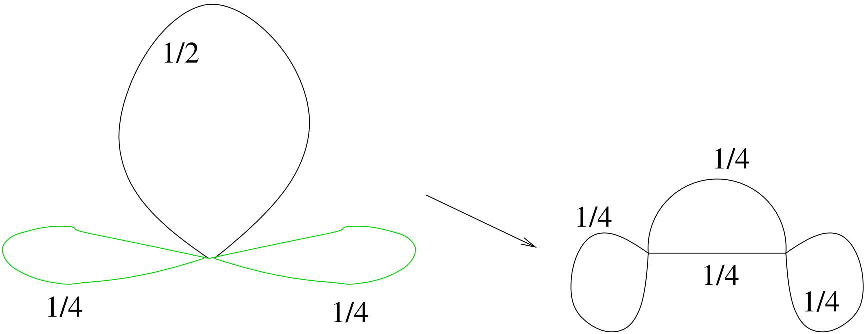
Let $f : S \to S'$ be a homotopy equivalence between two closed hyperbolic surfaces that minimizes the Lipschitz constant in its homotopy class. Then there is a geodesic lamination $\Lambda \subset S$ so that f is linear along the leaves of Λ with slope equal to the maximum. Moreover, f can be perturbed so that in the complement of Λ the Lipschitz constant is smaller than maximal.



Lipschitz metric on Outer space

Theorem

Let $f : \Gamma \to \Gamma'$ be a homotopy equivalence between two points of X_n that minimizes the Lipschitz constant in its homotopy class. Then there is a subgraph $\Gamma_0 \subset \Gamma$ so that f is linear along the edges of Γ_0 with slope equal to the maximum and Γ_0 has a train-track structure with at least two gates at each vertex. Moreover, f can be perturbed so that in the complement of Γ_0 the Lipschitz constant is smaller than maximal.



Lipschitz metric on Outer space

An application:

Theorem (B-Handel, 1992)

Every irreducible automorphism ϕ of F_n has a train-track representative.

Definition

An automorphism of F_n is reducible if it can be represented as a map $f : \Gamma \to \Gamma$ so that $f(\Gamma_0) \subseteq \Gamma_0$ for some proper subgraph with non-contractible components. Otherwise, it is irreducible.

Examples

$$a \mapsto ab$$

$$b \mapsto bab$$

$$c \mapsto c[a, b]$$

$$a \mapsto b, b \mapsto a$$

Train track maps

Definition

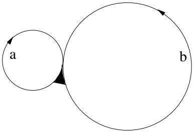
A map $f : \Gamma \to \Gamma$ is a train track map if it sends vertices to vertices and for every edge e and $k \geq 1$ the map $f^k|e$ is an immersion.

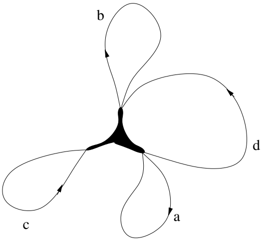
Definition

A map $f : \Gamma \to \Gamma$ is a train track map if it sends vertices to vertices, edges to immersed edge-paths, and the set of directions at every vertex can be divided into equivalence classes (gates) so that

- $d_1 \not\sim d_2 \Rightarrow f(d_1) \not\sim f(d_2)$, and
- for every edge e every turn in $f(e)$ consists of inequivalent directions.

Train track maps





$$a \mapsto ab$$

$$b \mapsto bab$$

$$|a| = 1, |b| = \lambda - 1$$
$$\lambda^2 - 3\lambda + 1 = 0$$

$$a \mapsto b$$

$$b \mapsto c$$

$$c \mapsto dA$$

$$d \mapsto DC$$

$$|a| = 1, |b| = \lambda$$
$$|c| = \lambda^2, |d| = \lambda^3 - 1$$
$$\lambda^4 - \lambda^3 - \lambda^2 - \lambda + 1 = 0$$

Train track maps

Train tracks are good for understanding dynamics. For example, assuming f is a train track representative of ϕ.

- growth rates: $|f^i(w)| \sim \lambda^i$ for "most" w. We write $\lambda = GR(\phi)$.
- fixed subgroups: $rank(Fix\ f_*) \leq n$ (and most of the time it is trivial).
- dynamics at infinity.

Proof of existence of train tracks

Proof.

(Sketch) Parallel to Bers' proof of Nielsen-Thurston classification. Consider

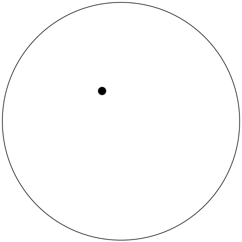
$$\Phi : X_n \to [0, \infty)$$

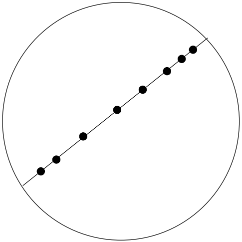
$$\Phi(\Gamma) = d(\Gamma, \phi(\Gamma))$$

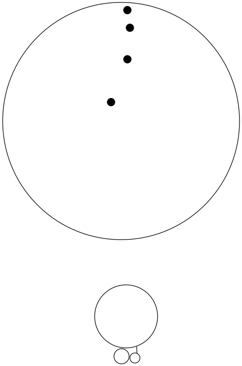
There are 3 cases:

- $\inf \Phi = 0$ and is realized. Then there is Γ with $\phi(\Gamma) = \Gamma)$ so ϕ has finite order.
- $\inf \Phi > 0$ and is realized, say at Γ. Apply above Theorem to $\phi : \Gamma \to \phi(\Gamma)$. Argue that $\Gamma_0 = \Gamma$ or else ϕ is reducible. Train-track structure on Γ_0 can be promoted to give the theorem.
- $d = \inf \Phi$ is not realized. Let $\Gamma_i \in X_n$ have $d(\Gamma_i, \phi(\Gamma_i)) \to d$. Argue that projections to $X_n/Out(F_n)$ leave every compact set. Thus Γ_i has "thin part" which must be invariant, so ϕ is reducible.

Proof of existence of train tracks





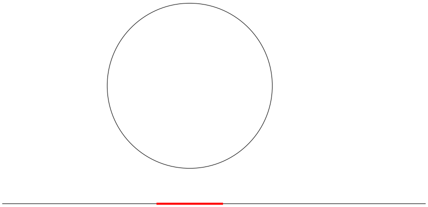


Axes

This proof also shows that irreducible ϕ has an axis with translation length $\log \lambda$, where λ is the expansion rate of ϕ.

Theorem (Yael Algom-Kfir, 2008)

Axes of irreducible elements are strongly contracting, i.e. the projection of any ball disjoint from the axis to the axis has uniformly bounded size.



The analogous theorem in Teichmüller space was proved by Minsky (1996).

Corollary (Yael Algom-Kfir)

Fully irreducible elements in $Out(F_n)$ are Morse.

Intersection numbers

If T_1, T_2 are G-trees, Scott-Swarup defined the intersection number $I(T_1, T_2)$.
When T_1, T_2 are dual to scc's (or measured laminations) on a surface, one gets the usual intersection number.
Guirardel reinterpreted $I(T_1, T_2)$: there is a canonical G-invariant subset $C \subset T_1 \times T_2$ ("Guirardel core") and

$$I(T_1, T_2) = vol(C/G)$$

Intersection numbers

Theorem (Thurston)

If f is a pseudo-Anosov homeomorphism then $I(a, f^n(a)) \sim \lambda^n$ for any scc a, where $\lambda = GR(f)$.

Theorem (Behrstock-B-Clay)

If f is a fully irreducible automorphism of F_n then

$$I(T, f^n(T)) \sim \max\{\lambda, \mu\}^n \text{ or } n\lambda^n$$

where $\lambda = GR(f)$ and $\mu = GR(f^{-1})$.

Constructing fully irreducible automorphisms

Theorem (Thurston 1979, Penner 1988, Hamidi-Tehrani 2002)

Let a, b be two scc's that fill a closed hyperbolic surface. Then $< D_a, D_b >$ is free and any element not conjugate to a power of D_a or D_b is pseudo-Anosov.

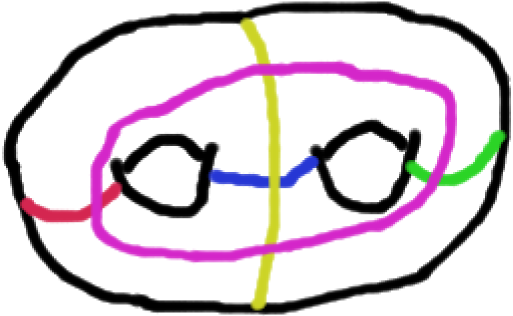
Theorem (Clay-Pettet, in preparation)

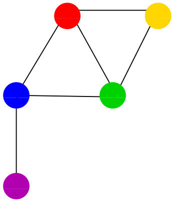
Let S, T be two cyclic F_n-trees that fill F_n. Then for large $k > 0$ the group $< D_S^k, D_T^k >$ is free and any element not conjugate to a power of D_S or D_T is fully irreducible.

Curve complex

Definition (Harvey)

Let Σ be a compact surface. The curve complex $\mathcal{C}(\Sigma)$ is the simplicial complex whose vertices are isotopy classes of essential nonperipheral simple closed curves in Σ and simplices correspond to collections that can be represented by pairwise disjoint curves.





Curve complex

We would like to have an analog of the following:

Theorem (Masur-Minsky, 1999)

The curve complex is δ-hyperbolic.

The candidates for the curve complex are:

- the splitting complex, and
- the free factor complex.

Conjecture

They are both δ-hyperbolic.

Curve complex

There are constructions of δ-hyperbolic $Out(F_n)$-graphs (or even quasi-trees) [B-Feighn 2008, B-Bromberg-Fujiwara 2009], but they depend on some choices. They are good enough for some applications:

- $H_b^2(Out(F_n); \mathbb{R})$ is infinite-dimensional.
- There are arbitrarily large balls in the Cayley graph of $Out(F_n)$ consisting of fully irreducible automorphisms.