Exclusion processes: some recent results and open questions

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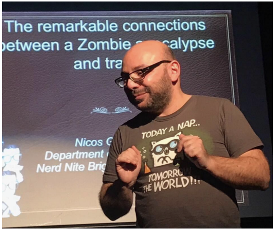
Interacting Particle Systems and Hydrodynamic Limits, Montréal
March 2022

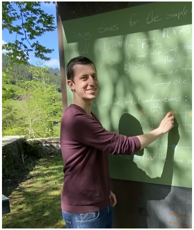
Coauthors

I am exploring exclusion processes with



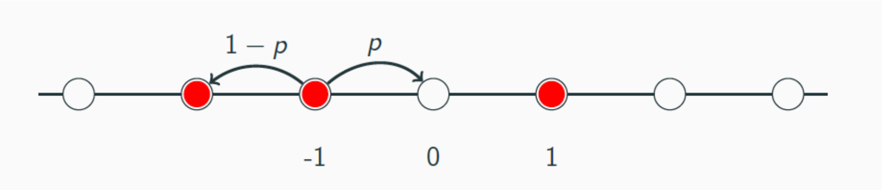
Nicos Georgiou

Evita Nestoridi



Dominik Schmid

Exclusion process on the line



Particles are equipped with rate 1 Poisson clocks: when a clock rings, the particle moves to the right with prob p and to the left with prob. $1-p$, provided the target site is empty.

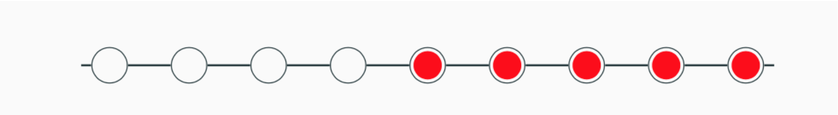
Invariant measures

Well-known that for all $p \in [0, 1]$, the Bernoulli product measures ν_ρ with parameter ρ (the sites are independently occupied with prob. ρ and empty with prob. $1 - \rho$) are invariant, for all $\rho \in [0, 1]$.
In the symmetric case $p = \frac{1}{2}$, these are all the (extremal) invariant distributions.

If $p > \frac{1}{2}$, there are further non-translation invariant distributions, the "blocking measures". They are concentrated on configurations with only finitely many particles to the left of the origin, and only finitely many empty sites to the right of the origin.

Example

If $p = 1$ - call this case TASEP for "Totally Asymmetric Exclusion Process" - for all N, the configurations with empty sites to the left of N and occupied sites to the right of N are fixed points.



What happens if the process does not start from an invariant distribution?

Consider the product measure $\nu_{\lambda,\rho}$ on $\{0,1\}^{\mathbb{Z}}$ with marginals

$$\nu_{\lambda,\rho}(\{\eta : \eta(x) = 1\}) = \begin{cases} \lambda & \text{if } x < 0 \\ \rho & \text{if } x \geq 0. \end{cases}$$

Let $\nu_{\lambda,\rho} T_t$ denote the distribution of the process at time t, when started from $\nu_{\lambda,\rho}$.

Theorem

In the symmetric case, i.e. if $p = \frac{1}{2}$,

$$\lim_{t\to\infty} \nu_{\lambda,\rho} T_t = \nu_{\frac{\lambda+\rho}{2}}$$

(recall ν_γ is the Bernoulli product measure with parameter γ).

Heuristic argument: Let μ_t be the distribution of the process at time t and

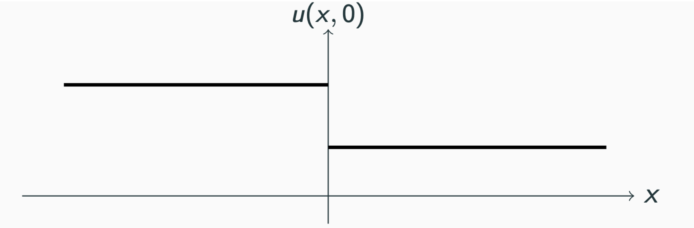
$$u(x,t) = \mu_t(\{\eta : \eta(x) = 1\}).$$

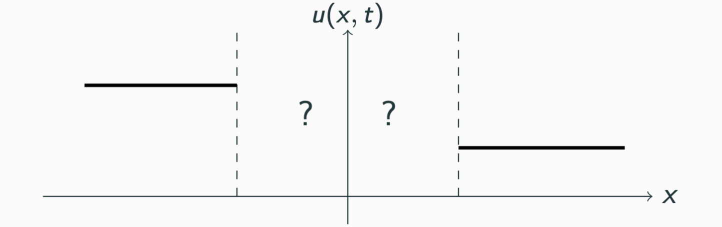
If $p = \frac{1}{2}$, then

$$\frac{d}{dt}u(x,t) = \frac{1}{2}u(x-1,t) + \frac{1}{2}u(x+1,t) - u(x,t)\,.$$

This is a discrete version of the heat equation

$$\frac{\partial u}{dt} = \frac{1}{2}\frac{\partial^2 u}{dx^2}\,.$$


u(x, 0)
x


u(x, t)
?
?
x

In the asymmetric case, the situation is more interesting.

Theorem

Assume $p > \frac{1}{2}$. Then

$$\lim_{t\to\infty} \nu_{\lambda,\rho} T_t = \begin{cases} \nu_{\frac{1}{2}} & \text{if } \lambda \geq \frac{1}{2} \text{ and } \rho \leq \frac{1}{2}, \\ \nu_\rho & \text{if } \rho \geq \frac{1}{2} \text{ and } \lambda + \rho > 1, \\ \nu_\lambda & \text{if } \lambda \leq \frac{1}{2} \text{ and } \lambda + \rho < 1, \\ \frac{1}{2}\nu_\lambda + \frac{1}{2}\nu_\rho & \text{if } 0 < \lambda < \rho \text{ and } \lambda + \rho = 1. \end{cases}$$

Heuristic argument: as before, let μ_t be the distribution of the process at time t and

$$u(x,t) = \mu_t(\{\eta : \eta(x) = 1\}).$$

If $p > \frac{1}{2}$ and $q = 1 - p$, then $u(x,t)$ solves a discrete version of Burgers' equation

$$\frac{\partial u}{dt} - (p-q)\frac{\partial}{\partial x}\left[u(1-u\right] = 0\,. \tag{1}$$

The initial condition should be

$$u(x,t) = \begin{cases} \lambda & \text{if } x < 0 \\ \rho & \text{if } x \geq 0 \end{cases}$$

but this is not a continuous function.

$$u(x,t) = \begin{cases} \lambda & \text{if } x \le c_1 t \\ a(t)x + b(t) & \text{if } c_1 t < x < c_2 t \\ \rho & \text{if } x \ge c_2 t \end{cases}$$

where $c_1 < c_2$ and $a(t), b(t)$ such that u is continuous, i.e.

$$a(t) = \frac{\rho - \lambda}{(c_2 - c_1)t} \qquad b(t) = \frac{\lambda c_2 - \rho c_1}{c_2 - c_1}$$

Substituting into (1) gives

$$c_1 = (p-q)(1-2\lambda), \qquad c_2 = (p-q)(1-2\rho).$$

If $\lambda > \rho$, $c_1 < c_2$ and this IS the solution, and the discontinuity is smoothed out. If $\lambda < \rho$, this does NOT yield a solution and the (true) solution is

$$u(x,t) = u(x - vt, 0) \text{ with } v = (p-q)(1-\lambda-\rho)\,.$$

Traveling wave: shape of the solution does not change over time, but it moves at speed v.

Second class particles

If $\lambda \leq \rho$, v is the speed of a *second class particle*. Second class particles are "between" particles and empty sites. Define the exclusion process as follows:

- empty sites have less priority than second class particles, second class particles have less priority than first class particles
- update the edges: exchange values at x and at $x+1$ with prob. p if $\eta(x)$ has higher priority than $\eta(x-1)$ and with prob. $1-p$ otherwise.

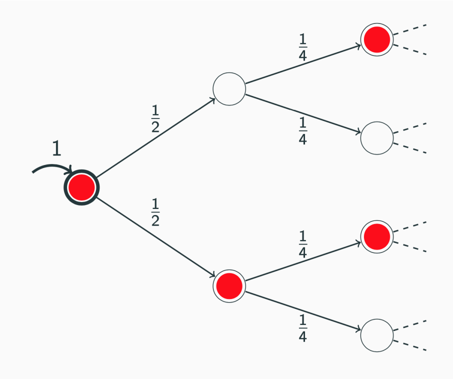
Consider the measure $\nu_{\lambda,\rho}$ and put a second class particle at the origin. Denote by X_t the location of the second class particle at time t. Have

$$\frac{X_t}{t} \to v \text{ almost surely for } t \to \infty.$$

.

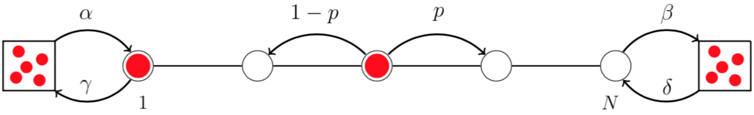
Open question: general graphs?

Take a regular tree with a reservoir at the root.
Conjecture: the limiting measure, when starting from the empty tree, is a Bernoulli product measure with parameter $\frac{1}{2}$.



A finite system: exclusion processes with open boundaries

Consider an exclusion process on the segment $[1, \ldots, N]$ where particles can enter and leave at the boundaries.



If $\min(\alpha, \beta, \gamma, \delta) > 0$, the process is an irreducible Markov chain with finite state space $\{0,1\}^N$, hence it converges to its unique invariant distribution $\mu = \mu_N$.
How fast is the convergence if N is large?

Mixing times

For a sequence of Markov chains $(\eta_t^N)_{t\geq 0}$ with state spaces Ω_N and invariant distributions μ_N, define

- Total variation distance:

$$||\mathbb{P}_\eta(\eta_t \in \cdot) - \mu_N||_{TV} = \frac{1}{2}\sum_{\xi\in\Omega_N} |P_\eta(\eta_t = \xi) - \mu_N(\xi)|$$

- Maximal distance to equilibrium at time t:

$$d(t) = \max_{\eta\in\Omega_N} ||\mathbb{P}_\eta(\eta_t \in \cdot) - \mu_N||_{TV}$$

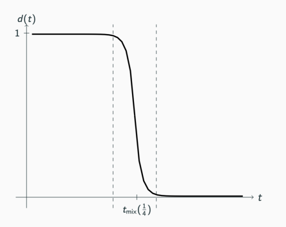
- ε-mixing time (for $\varepsilon \in (0,1)$):

$$t_{\text{mix}}^N(\varepsilon) = \inf\{t \geq 0 : d(t) < \varepsilon\}$$

Cutoff phenomenon

A sequence of Markov chains $(\eta_t^N)_{t\geq 0}$ satisfies *cutoff* if the distance to equilibrium drops from near 1 to near 0 over a time interval which is asymptotically smaller than the mixing time, i.e.

$$\lim_{N\to\infty} \frac{t^N_{\mathrm{mix}}(\varepsilon)}{t^N_{\mathrm{mix}}(1-\varepsilon)} = 1\,.$$



Relation with spectral gap

It was conjectured that the following “product condition” is equivalent to cutoff

$$\lim_{N\to\infty} \lambda_N \cdot t^N_{\mathrm{mix}}(\varepsilon) = \infty \tag{2}$$

where λ_N denotes the spectral gap of $(\eta^N_t)_{t\geq 0}$.
Turned out that (2) is necessary but not sufficient for cutoff. Nonetheless, (2) is sufficent for many models as for instance birth-and death chains, simple random walks on trees, exclusion processes which are reversible with respect to a product measure, ...

How does $t_{\mathrm{mix}}^{N}(\varepsilon)$ grow with N?

There are different regimes and we have various results for the growth of $t_{\mathrm{mix}}^{N}(\varepsilon)$, extending earlier results for closed boundaries (i.e. $\alpha = \beta = \gamma = \delta = 0$) by Hubert Lacoin and by David Wilson (symmetric case), by Itai Benjamini, Noam Berger, Christopher Hoffman, Elchanan Mossel and by Cyril Labbé, Hubert Lacoin (asymmetric case).

Symmetric, two-sided case

Theorem (NG, Evita Nestoridi, Dominik Schmid)

For $p = \frac{1}{2}$, the ε-mixing time of the simple exclusion process with open boundaries satisfies

$$\frac{1}{\pi^2} \leq \liminf_{N\to\infty} \frac{t_{\text{mix}}^N(\varepsilon)}{N^2 \log N} \leq \limsup_{N\to\infty} \frac{t_{\text{mix}}^N(\varepsilon)}{N^2 \log N} \leq C$$

for all $\varepsilon \in (0,1)$ and some constant $C = C(\alpha, \beta, \gamma, \delta)$.

Symmetric, one-sided case

Theorem (NG, Evita Nestoridi, Dominik Schmid)

For $p = \frac{1}{2}$, suppose that $\max(\alpha, \gamma) = 0$ and $\min(\beta, \delta) > 0$ holds. Then for all $\varepsilon \in (0, 1)$, the ε-mixing time of the simple exclusion process with open boundaries satisfies

$$\lim_{N \to \infty} \frac{t^N_{\text{mix}}(\varepsilon)}{N^2 \log N} = \frac{4}{\pi^2} .$$

In particular, cutoff occurs.

Substantial generalization by Justin Salez, "Universality of cutoff for exclusion with reservoirs": cutoff occurs if the process is reversible with respect to a product measure.

Current

Define the current J_t^N as follows. Assume $p \in (\frac{1}{2}, 1], \min(\alpha, \beta) > 0$.
Let J_t^{N+} be the number of particles which have entered at the left side until time t,
J_t^{N-} be number of particles which have exited at the left side until time t and, for $t \geq 0$,

$$J_t^N = J_t^{N+} - J_t^{N-}.$$

Lemma (Masaru Uchiyama, Tomohiro Sasamoto, Miki Wadati)

There are $a = a(\alpha, \gamma, p)$ *and* $b = b(\beta, \delta, p)$ *such that, with*

$$J = J(a,b,p) := \begin{cases} (2p-1)\frac{a}{(1+a)^2} & \text{if } a > \max(b,1) \text{ ``low density phase''} \\ (2p-1)\frac{b}{(1+b)^2} & \text{if } b > \max(a,1) \text{ ``high density phase''} \\ (2p-1)\frac{1}{4} & \text{if } \max(a,b) \le 1 \text{ ``max current phase''} \end{cases}$$

the current $(J_t^N)_{t \geq 0}$ *of the simple exclusion process with open boundaries satisfies*

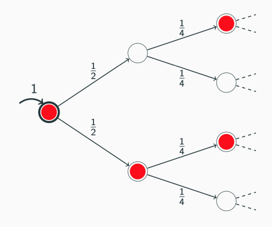
$$\lim_{t\to\infty} \frac{J_t^N}{t} = J_N$$

almost surely for some deterministic sequence $(J_N)_{N\in\mathbb{N}}$ *with* $\lim_{N\to\infty} J_N = J$.

Intuition: our process interpolates between two Bernoulli product measures with parameters $\frac{1}{1+a}$ and $\frac{b}{1+b}$, respectively.

Open question: general graphs?

Take a regular tree with a reservoir at the root.



Some results on the current (depending on the transition rates) in the paper "The TASEP on Galton-Watson trees", NG, Nicos Georgiou, Dominik Schmid.

Theorem (NG, Evita Nestoridi, Dominik Schmid)

For parameters $\alpha, \beta, \gamma, \delta \geq 0$ and $p > \frac{1}{2}$, suppose we are in the low density phase. Then there exists a constant $C_\ell = C_\ell(a, b, p) > 0$ such that the ε-mixing time of the simple exclusion process with open boundaries satisfies

$$\frac{1}{2p-1} \leq \liminf_{N\to\infty} \frac{t^N_{\text{mix}}(\varepsilon)}{N} \leq \limsup_{N\to\infty} \frac{t^N_{\text{mix}}(\varepsilon)}{N} \leq C_\ell$$

for all $\varepsilon \in (0,1)$. A similar statement holds for the high-density phase.

Theorem (Dor Elboim, Dominik Schmid, in progress)

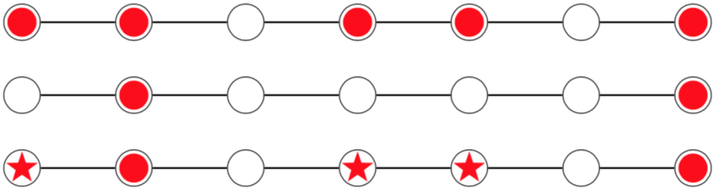
For parameters $\alpha, \beta, \gamma, \delta \geq 0$ *and* $p > \frac{1}{2}$, *suppose we are in the low density phase. Then there exists an (explicit) constant* $C = C(a, b, p) > 0$ *such that the* ε*-mixing time of the simple exclusion process with open boundaries satisfies*

$$\liminf_{N \to \infty} \frac{t_{\text{mix}}^N(\varepsilon)}{N} = C$$

for all $\varepsilon \in (0, 1)$. *A similar statement holds for the high-density phase. In particular, cutoff occurs!*

Important ingredient of the proof: Second class particles

Second class particles are “between” particles and empty sites. Crucial fact: if you couple two (ordered) processes, the disagreements move as second class particles!



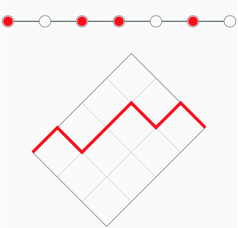
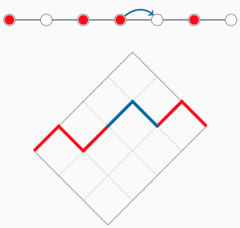
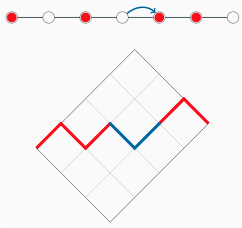
Conjecture for the max. current phase

Conjecture

When $\max(a, b) \leq 1$ *holds, i.e. in the maximal current phase, we have that the* ε*-mixing time of the simple exclusion process with open boundaries is of order* $N^{3/2}$ *for all* $\varepsilon \in (0, 1)$*. Moreover, the cutoff phenomenon does not occur.*

In the case $p = 1$, this has been partially proved by Dominik Schmid, see "Mixing times for the TASEP in the maximal current phase", Preprint 2021, and Dominik Schmid and Allan Sly, see "Mixing times for the TASEP on the circle", Preprint 2022.

Height function representation

Fluctuations of the height functions fall in different universality classes!
See recent work of Ivan Corwin, Alisa Knizel, Shalin Parekh and others.

Thanks for your attention!