Duality relations among particle systems, diffusions and redistribution models with an open boundary

Chiara Franceschini

University of Modena and Reggio Emilia

In progress work with R. Frassek and C. Giardinà

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Introduction

- Motivation comes from **non-equilibrium statistical mechanics**: open processes with boundary reservoirs: $L = L_{left} + L_{bulk} + L_{right}$.
- **Goal:** the full microscopic description of the non-equilibrium stationary state.
- Duality provides a link between **interacting particle systems** and **quantum spin chains**.
- These models are related to the **Kipnis-Marchioro-Presutti model** (1982), a mathematical model for energy transport.

Duality definition

Definition

Let $(z_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$ Markov process on Ω and Ω^{dual} with generators L and L^{dual}, respectively. z_t is **dual** to ξ_t with duality function $D : \Omega \times \Omega^{dual} \to \mathbb{R}$ if $\forall t \geq 0$,

$$\mathbb{E}_z(D(z_t, \xi)) = \mathbb{E}_\xi(D(z, \xi_t)) \qquad \forall (z, \xi) \in \Omega \times \Omega^{dual} \ .$$

Equivalently,

$$LD(\cdot, \xi)(z) = L^{dual} D(z, \cdot)(\xi)$$

Duality enables to connect, via a duality function, the process of interest to another one which is a **simpler** process.

Boundary driven context for interacting particle systems

The process generator reads

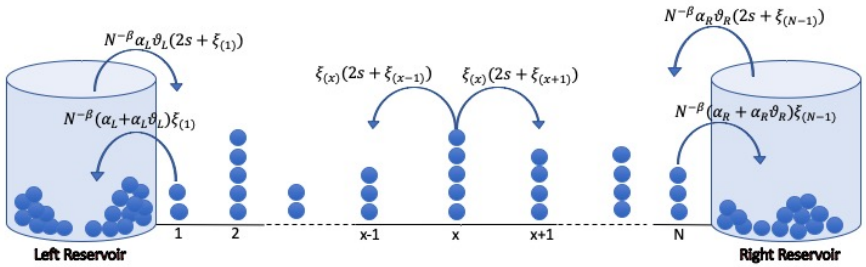
$$L^{SIP(2s)} = \frac{1}{N^\beta} L_{left} + \sum_{x=1}^{N-1} L_{x,x+1} + \frac{1}{N^\beta} L_{right} \quad \text{where}$$

$$\begin{aligned} L_{x,x+1} f(\xi) &= \xi_x(2s+\xi_{x+1}) \left[f(\xi^{x,x+1}) - f(\xi) \right] \\ &\quad + \xi_{x+1}(2s+\xi_x) \left[f(\xi^{x+1,x}) - f(\xi) \right] \\ L_{left} f(\xi) &= \alpha_L \vartheta_L (2s + \xi_1) \left[f(\xi^{0,1}) - f(\xi) \right] \\ &\quad + (\alpha_L + \alpha_L \vartheta_L) \xi_1 \left[f(\xi^{1,0}) - f(\xi) \right] \end{aligned}$$

Boundary rates scale with the size of the system (think of $\beta = 0$ for simplicity).

- State space $\mathbb{N}_0^N$
- $\xi_x = \sharp$ particles at x
- $\xi^{x,x+1} = \xi - \delta_x + \delta_{x+1}$
- Each site can accommodate an arbitrary number of particles
- For $s \to 0$ it exhibits condensation phenomena (Grosskinsky, Redig, Vafayi 2011)

Hydrodynamics



Theorem (F., Gonçalves, Sau (2020))

After a diffusive space/time rescale the hydrodynamic equations for the inclusion process is the heat equations with diffusive coefficient 2s

$$\begin{cases} \partial_t \vartheta(t,u) &= 2s\,\partial_u^2 \vartheta(t,u) & t\in[0,\infty)\,,\ u\in(0,1) \\ \vartheta(0,u) &= \vartheta_0(u) & u\in[0,1]\,, \end{cases}$$

and appropriate boundary conditions depending on the reservoirs strength tuned by the parameter β.

L_{bulk} and two different representations of $\mathfrak{su}(1,1)$

$$L_{bulk} = \sum_{x=1}^{N} L_{x,x+1} = \sum_{x=1}^{N} K_x^+ K_{x+1}^- + K_x^- K_{x+1}^+ - 2K_x^0 K_{x+1}^0 + 2s^2, \qquad s \in \mathbb{R}_+$$

$$\begin{cases} K^+ = z \\ K^- = \left(z\partial_z^2 + 2s\partial_z\right) \\ K^0 = (z\partial_z + s) \end{cases} \qquad \begin{cases} K^+ f(\xi) = (\xi + 2s) f(\xi + 1) \\ K^- f(\xi) = \xi f(\xi - 1) \\ K^0 f(\xi) = (\xi + s) f(\xi) \end{cases}$$

Lie brackets are: $[K^0, K^\pm] = \pm K^\pm$ and $[K^-, K^+] = 2K^0$

- The first representation give rise to the bulk part of the **BEP(**$2s$**)**

$$L_{x,x+1}^{BEP} = z_x z_{x+1} \left(\partial_{z_x} - \partial_{z_{x+1}}\right)^2 - 2s(z_x - z_{x+1}) \left(\partial_{z_x} - \partial_{z_{x+1}}\right)$$

- The second representation give rise to the bulk part of the **SIP(**$2s$**)**

$$L_{x,x+1}^{SIP} f(\xi) = \xi_x(2s + \xi_{x+1}) \left[f(\xi^{x,x+1}) - f(\xi)\right] + \xi_{x+1}(2s + \xi_x) \left[f(\xi^{x+1,x}) - f(\xi)\right]$$

Duality for BEP and SIP

Theorem (Carinci, Giardinà, Giberti, Redig (2013))

L^{BEP} and L^{SIP} are dual with $D(z,\xi) = \prod_{x=1}^{N} \frac{z_x^{\xi_x}}{\Gamma(\xi_x + 2s)}$

Theorem (F., Giardinà (2017))

L^{BEP} and L^{SIP} are dual with $D(z,\xi) = \prod_{x=1}^{N} {}_1F_1\left(\begin{matrix} -\xi_x \\ 2s \end{matrix} \middle| z_x \right)$ *with a different* $\mathfrak{su}(1,1)$ *representation:*

$$\begin{cases} K^+ f(\xi) = (\xi + 2s) f(\xi + 1) + 2(\xi + s) f(\xi) - \xi f(\xi - 1) \\ K^- f(\xi) = \xi f(\xi - 1) \\ K^0 f(\xi) = (\xi + s) f(\xi) - \xi f(\xi - 1) \end{cases}$$

Connection with population genetics

Theorem (Carinci, Giardinà, Giberti, Redig (2015))

- *The BEP(2s) on the complete graph with d vertices and initial condition $\sum_{x=1}^{d} z_x = 1$ coincides with the $d-$types* **Wright-Fisher diffusion** *with parent-independent mutation $m = 2s(d-1)$*

- *The SIP(2s) on the complete graph with d vertices and initial condition $\sum_{x=1}^{d} \xi_x = N$ coincides with the $d-$types* **Moran model** *with population size N and parent-independent mutation $m = 2s(d-1)$*

Brownian Energy Process with an open boundary

The generator is $L^{BEP} = L_{left} + \sum_{x=1}^{N} L_{x,x+1} + L_{right}$ where

$$L_{x,x+1} = z_x z_{x+1} \left(\partial_{z_x} - \partial_{z_{x+1}}\right)^2 - 2s(z_x - z_{x+1})\left(\partial_{z_x} - \partial_{z_{x+1}}\right)$$

$$L_{left} = T_- \left(2s\partial_{z_1} + z_1 \partial_{z_1}^2\right) - \frac{1}{2} z_1 \partial_{z_1}$$

$$L_{right} = T_+ \left(2s\partial_{z_N} + z_N \partial_{z_N}^2\right) - \frac{1}{2} z_N \partial_{z_N}$$

for $z \in (\mathbb{R}^+)^N$, z_x represents the energy at site x

Set $s = 1/2$ + perform an instantaneous thermalization limit = KMP

For z_x and z_{x+1} independent Exponential $(\frac{1}{T})$, then

$$\frac{z_x}{z_x + z_{x+1}} \mid z_x + z_{x+1} = p \sim U([0,1])$$

The Kipnis-Marchioro-Presutti model (1982)

The generator is $L^{KMP} = L_{left} + \sum_{x=1}^{N} L_{x,x+1} + L_{right}$ where

$$L_{x,x+1} f(\mathbf{z}) = \int_0^1 dp \, [f(z_1, \ldots, p(z_x + z_{x+1}), (1-p)(z_x + z_{x+1}), \ldots z_N) - f(\mathbf{z})]$$

$$L_{left} f(\mathbf{z}) = \int_0^\infty dz_1' \frac{e^{-z_1'/T_-}}{T_-} [f(z_1', \ldots, z_N) - f(\mathbf{z})]$$

$$L_{right} f(\mathbf{z}) = \int_0^\infty dz_N' \frac{e^{-z_1'/T_+}}{T_+} [f(z_1, \ldots, z_N') - f(\mathbf{z})]$$

for $z \in (\mathbb{R}^+)^N$, z_x represents the energy at site x

After the redistribution: with $p \sim U([0,1])$ or in general $Beta(2s, 2s)$
$z_x' = p(z_x + z_{x+1})$ is the new energy at site x and
$z_{x+1}' = (1-p)(z_x + z_{x+1})$ is the new energy at site $x+1$

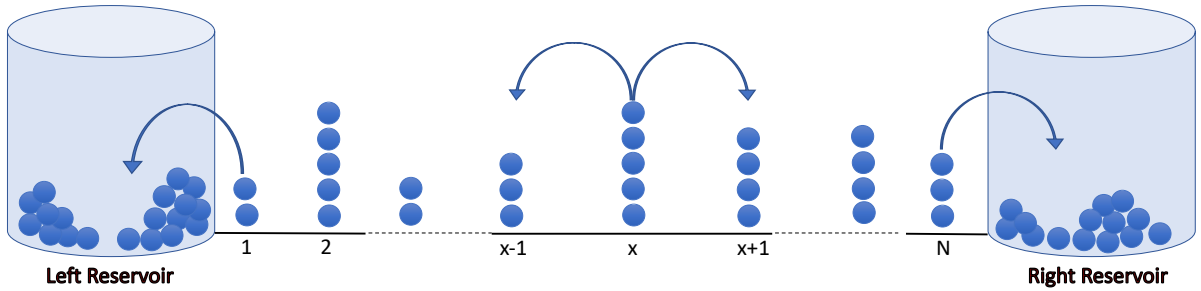
Duality: open boundary vs absorbing boundary

Theorem (Carinci, Giardinà, Giberti, Redig (2013))

L^{BEP} and L^{SIP} are dual with $D(z,\xi) = (T_-)^{\xi_0} \prod_{x=1}^{N} \frac{z_x^{\xi_x}}{\Gamma(\xi_x + 2s)} (T_+)^{\xi_N}$ *where*

$L^{SIP} = L^{SIP}_{left} + L^{SIP}_{bulk} + L^{SIP}_{right}$ *is an* **only absorbing** *particle system with*

$$L^{SIP}_{left} f(\xi) = \frac{\xi_1}{2}\left[f(\xi^{1,0}) - f(\xi)\right] \qquad \text{and} \qquad L^{SIP}_{right} f(\xi) = \frac{\xi_N}{2}\left[f(\xi^{N,N+1}) - f(\xi)\right]$$



Remark 1: The thermalization limit of SIP gives the dual of the KMP model with the same duality function

Remark 2: The dual processes are always absorbing in sites $x = 0$ and $x = N + 1$ which are **not** artificial sites

Remark 3: The moment duality function is not unique!

Theorem (Floreani, Redig, Sau (2022))

Proved that in the context of an open boundary the orthogonal dualities still holds.

Characterization of the stationary measure: equilibrium and non-equilibrium

$$E_{\mu_{ss}}[D(z,\xi)] = \sum_{m=0}^{k} T_-^m T_+^{k-m} P_\xi(m) \ ,$$

- $k = \sum_{x=0}^{N} \xi_x$ is the total number of dual particles
- $P_\xi(m)$ is the probability that m particles are absorbed at the left reservoir (and the remaining $k-m$ go to the right reservoir) starting from the configuration ξ.

$$E_{\mu_{ss}}[D(z,\xi)] = \lim_{t\to\infty} E_z[D(z_t,\xi)] =$$

$$\lim_{t\to\infty} E_\xi[D(z,\xi_t)] = E_\xi\left[T_-^{\xi_\infty(0)} T_+^{\xi_\infty(N+1)}\right] =$$

$$\sum_{m=0}^{k} T_-^m T_+^{k-m} P_\xi(\xi_\infty(0) = m,\ \xi_\infty(N+1) = k-m) \ .$$

Equilibrium: $T_- = T_+ = T$

- Choose $\xi = k\delta_x$, then $D(z, k\delta_x) = \dfrac{z_x^k \Gamma(2s)}{\Gamma(2s+k)}$ and

$$E_{\mu_{ss}}[D(z, k\delta_x)] = E_{\mu_{ss}}\left[\frac{z_x^k \Gamma(2s)}{\Gamma(2s+k)}\right] = \sum_{m=0}^{k} T_-^m T_+^{k-m} P_\xi(m) = T^k$$

  implies

$$E_{\mu_{ss}} z_x^k = \frac{\Gamma(2s+k)}{\Gamma(2s)} T^k \,, \quad \text{namely} \quad z_x \sim Gamma(2s, T)$$

- In particular,

$$E_{\mu_{ss}}\left[\prod_{x=1}^{N} z_x^{\xi_x}\right] = \prod_{x=1}^{N} \frac{\Gamma(2s+\xi_x)}{\Gamma(2s)} T^{\xi_x}$$

Non-equilibrium: $T_- \neq T_+$

- Choose $\xi = \delta_x$, then $D(z, \delta_x) = \dfrac{z_x}{2s}$ and

$$E_{\mu_{ss}}\left[\frac{z_x}{2s}\right] = T_+ P_{\delta_x}(0) + T_- P_{\delta_x}(1) = T_- + (T_+ - T_-) P_{\delta_x}(0)$$
$$= T_- + (T_+ - T_-)\frac{x}{N+1}$$

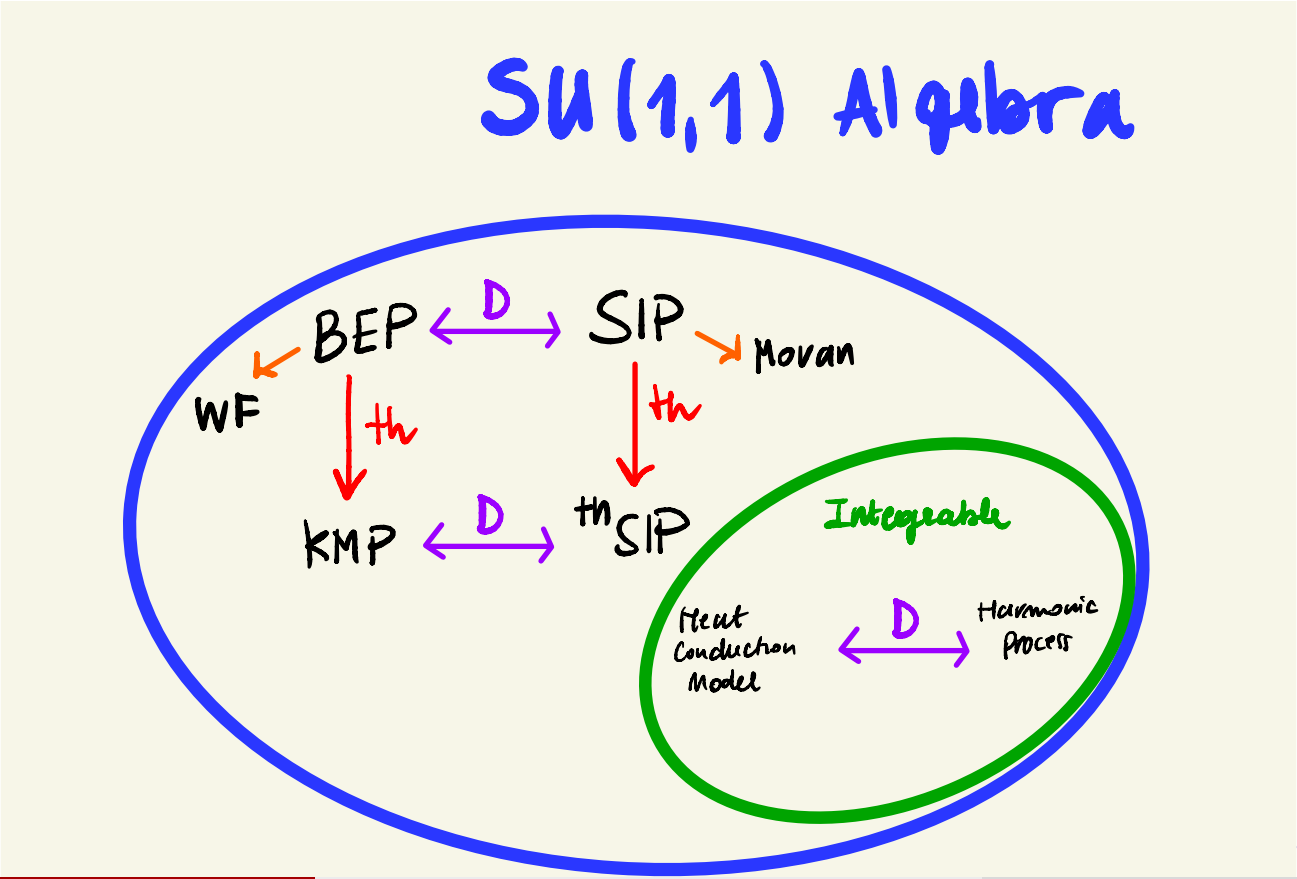
- Choose $\xi = \delta_x + \delta_y$, then $D(z, \delta_x + \delta_y) = \dfrac{z_x}{2s}\dfrac{z_y}{2s}$ and

$$E_{\mu_{ss}}\left[\frac{z_x}{2s}\frac{z_y}{2s}\right] = T_+^2 P_{\delta_x+\delta_y}(0) + T_+ T_- P_{\delta_x+\delta_y}(1) + T_-^2 P_{\delta_x+\delta_y}(2)$$

Duality and Integrability

The full microscopic description of the non-equilibrium stationary state remained not accessible for the KMP and BEP models as the absorption probabilities of the dual particles are not know in closed-form.

Only known up to second moments



Integrable heat conduction model

For $s = 1/2$ the generator is

$$\mathscr{L} = \mathscr{L}_1 + \sum_{x=1}^{N-1} \mathscr{L}_{x,x+1} + \mathscr{L}_N \qquad \text{where}$$

$$\mathscr{L}_{x,x+1} f(y) = \int_0^{y_x} \frac{d\alpha}{\alpha} \left[f(y_1, \dots, y_x - \alpha, y_{x+1} + \alpha, \dots, y_N) - f(y) \right] + \int_0^{y_{x+1}} \frac{d\alpha}{\alpha} \left[f(y_1, \dots, y_x + \alpha, y_{x+1} - \alpha, \dots, y_N) - f(y) \right]$$

$$\mathscr{L}_1 f(y) = \int_0^{y_1} \frac{d\alpha}{\alpha} \left[f(y_1 - \alpha, \dots, y_N) - f(y) \right] + \int_0^{+\infty} \frac{d\alpha}{\alpha} e^{-\alpha/T_-} \left[f(y_1 + \alpha, \dots, y_N) - f(y) \right]$$

After the redistribution: with $u \sim Beta\,(0,1)$ or in general $Beta\,(0, 2s)$
$y'_x = y_x - u y_x$ is the new energy at site x and
$y'_{x+1} = y_{x+1} + u y_x$ is the new energy at site $x+1$

The associated dual process: harmonic process

For $s = 1/2$ dual generator is

$$\mathscr{L}^{\mathrm{dual}} = \mathscr{L}_1^{\mathrm{dual}} + \sum_{x=1}^{N-1} \mathscr{L}_{x,x+1}^{\mathrm{dual}} + \mathscr{L}_N^{\mathrm{dual}} \qquad \text{where}$$

$$\mathscr{L}_{x,x+1}^{\mathrm{dual}} f(\xi) = \sum_{k=1}^{\xi_x} \frac{1}{k} \Big[f(\xi - k\delta_x + k\delta_{x+1}) - f(\xi) \Big] + \sum_{k=1}^{\xi_{x+1}} \frac{1}{k} \Big[f(\xi + k\delta_x - k\delta_{x+1}) - f(\xi) \Big]$$

and

$$\mathscr{L}_1^{\mathrm{dual}} f(\xi) = \sum_{k=1}^{\xi_1} \frac{1}{k} \Big[f(\xi - k\delta_1 + k\delta_0) - f(\xi) \Big]$$

In general the rate is $\varphi_s(k,n) = \frac{1}{k} \frac{\Gamma(n+1)\Gamma(n-k+2s)}{\Gamma(n+2s)\Gamma(n-k+1)}$

Relation to previous models of L_{bulk}

For $s = 1/2$, the asymmetric version is the MADAM model [Sasamoto, Wadati (1998)]:

$$L^{MADAM} f(\xi) = \sum_{k=1}^{\xi_x} \frac{1}{[k]_q} \left[f(\xi - k\delta_x + k\delta_{x+1}) - f(\xi) \right] + \sum_{k=1}^{\xi_{x+1}} \frac{q^k}{[k]_q} \left[f(\xi + k\delta_x - k\delta_{x+1}) - f(\xi) \right]$$

where $[k]_q := \frac{1-q^k}{1-q} \to 1$ as $q \to 1$

For general spin s it is a $q-$Hahn type model [Barraquand, Corwin (2014)]

The absorbing harmonic is also the dual of an open boundary harmonic [Frassek, Giardinà, Kurchan (2020)]

Equilibrium and non-equilibrium steady states

Theorem (F., Frassek, Giardinà (2022+))

- *For $T_- = T_+ = T$ the unique stationary measure is* **homogeneous product of Gamma distributions** *of rate parameter T and shape parameter $2s$.*
- *For $T_- \neq T_+$ we can characterize all moments of the non-equilibrium stationary measure μ_{ss}:*

$$\int_{\mathbb{R}_+} \prod_{x=1}^{N} \frac{y_x^{\xi_x}\Gamma(2s)}{\Gamma(2s+\xi_x)}\mu_{ss}(dy) = \sum_{n=0}^{|\xi|} T_+^{|\xi|-n}(T_- - T_+)^n g_\xi(n)$$

where

$$g_\xi(n) = \sum_{1\leq i_1<\ldots<i_n\leq|\xi|} \prod_{j=1}^{n} \frac{2s(N+1-x_{i_j})+n-j}{2s(N+1)+n-j}$$

First correlations

- One point correlations: let $|\xi| = 1$ and $\xi = \delta_{x_1}$ then $\mathbb{E}_{\mu_{ss}}\left[\frac{y_{x_1}}{2s}\right] = T_+ g_{x_1}(0) + (T_- - T_+) g_{x_1}(1)$ where $g_{x_1}(0) = 1$ and $g_{x_1}(1) = 1 - \frac{x}{N+1}$ and so

$$\mathbb{E}_{\mu_{ss}}\left[\frac{y_{x_1}}{2s}\right] = T_+ + (T_- - T_+)\left(1 - \frac{x}{N+1}\right)$$

- Two point correlations: let $|\xi| = 2$
  - If $\xi = \delta_{x_1} + \delta_{x_2}$ then

$$\mathbb{E}_{\mu_{ss}}\left[\frac{y_{x_1}}{2s}\frac{y_{x_2}}{2s}\right] = T_+^2 + T_+(T_- - T_+)\left(1 - \frac{x_1}{N+1} + 1 - \frac{x_2}{N+1}\right) + (T_- - T_+)^2\left(1 - \frac{2sx_1}{2s(N+1)+1}\right)\left(1 - \frac{x_2}{N+1}\right)$$

  - If $\xi = 2\delta_{x_1}$ then

$$\mathbb{E}_{\mu_{ss}}\left[\frac{y_{x_1}^2}{2s(2s+1)}\right] = \text{as above with } x_1 = x_2$$

Variance and Covariance

- The covariance is

$$\mathbb{E}_{\mu_{ss}}\left[y_{x_1} y_{x_2}\right] - \mathbb{E}_{\mu_{ss}}\left[y_{x_1}\right]\mathbb{E}_{\mu_{ss}}\left[y_{x_2}\right] = (2s)^2(T_- - T_+)^2 \frac{x_1}{(N+1)^2}\frac{N+1-x_2}{2s(N+1)+1}$$

Formal: setting $s = -1/2$ recover the correlations for the SSEP given by the matrix formulation method in Derrida, Lebowitz, Speer (2006)

- The variance is

$$\mathbb{E}_{\mu_{ss}}\left[y_{x_1}^2\right] - \mathbb{E}_{\mu_{ss}}\left[y_{x_1}\right]^2 = 2s\Bigg\{ T_+^2 + 2T_+(T_- - T_+)\left(1 - \frac{x_1}{N+1}\right)$$

$$(T_- - T_+)^2\left(1 - \frac{x_1}{(N+1)}\right)\left(1 - \frac{2sx_1N}{(2s(N+1)+1)(N+1)}\right)\Bigg\}$$

Summary

Theorem (Floreani, Redig, Sau (2022))

For the **non-integrable** *models, using the dual configuration* $\xi = \delta_{x_1} + \ldots + \delta_{x_n}$ *on any graph leads to*

$$\mathbb{E}_{\mu_{ss}} \prod_{i=1}^{n} \left(\frac{z_{x_i}}{2s} - T_{x_i} \right) = (T_- - T_+)^n f(\xi)$$

and conjectured that $f(\xi) \sim O(N^{-n})$ for the one dimensional chain.

For the **integrable** models a close formula for the $n-$points correlations can be written.

Summary

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Thanks for the attention :)