

LIMIT THEOREMS FOR EXPONENTIAL RANDOM GRAPHS

by Alessandro Bianchi - UNIVERSITA' DI PADOVA

[joint work with François Bolley and Elena Mariani]

INTERACTING PARTICLE SYSTEMS AND HYDRODYNAMIC LIMITS - MARCH 20-25 2022 - CRM MONTREAL

1. Setting

a. Exponential random graphs (see Park, Newman '04 for history)

We set * $\mathcal{G}_m := \{G \text{ simple graph on } [m]\}$, $G = (V, E)$

* $V = \{1, \dots, m\} =: [m]$, $m \in \mathbb{N}$

* $E \subseteq \mathcal{E}_m := \{(i, j) : i, j \in [m], i \neq j\}$

→ ExpRG defined by a Gibbs probability measure on \mathcal{G}_m

Edge-triangle model (Strauss's model)

For $\alpha, h \in \mathbb{R}$

$$P_{m;\alpha,h}(G) = \frac{e^{H_{m;\alpha,h}(G)}}{\sum_{G \in \mathcal{G}_m} e^{H_{m;\alpha,h}(G)}}, \quad \text{partition function}$$

Where $H_{m;\alpha,h}(G) = \frac{\alpha}{m} T(G) + h E(G)$

Remark: If $\alpha=0 \Rightarrow P_{m,0,h}(G) \propto e^{h E(G)} = P_{m,p}^{\text{ER}}(G) \quad p = \frac{e^h}{1+e^h}$

Equivalently:

$$G_m \xleftarrow[\text{sa}]{\sim} \{0,1\}^{\binom{E_m}{2}} \ni \underline{x} \rightarrow P_{m,\alpha,h}(\underline{x}) = e^{H_{m,\alpha,h}(\underline{x})}/Z_{m,\alpha,h}$$

where

$$H_{m,\alpha,h}(\underline{x}) = \frac{\alpha}{m} \sum_{\{(i,j,k) \in \binom{E_m}{3}\}} x_i x_j x_k + h \sum_{i \in E_m} x_i$$

→ set of triangles over $[m]$

Goal: Determine the limiting behavior of the edge density

$$\frac{2E(G)}{m^2} \in [0,1]_m = \left\{ 0, \frac{2}{m^2}, \dots, 1 - \frac{1}{m} \right\}$$

Equivalently: $\underline{X} = (X_i)_{i \in E_m}$ with law $P_{m,\alpha,h}$, $S_m := \sum_{i \in E_m} X_i$

$$2 \frac{S_m}{m^2} \quad \left(= \frac{S_m}{m^2/2} \right)$$

2. Relevant results on ExprG

a. Free energy: maximizers of $P_{m,\alpha,n}$ as $n \rightarrow \infty$

Let $f_m(\alpha, h) := \frac{1}{m^2} \ln Z_{m; \alpha, h}$ and $f(\alpha, h) = \lim_{m \rightarrow \infty} f_m(\alpha, h)$

Thm [Chatterjee, Diaconis '13]: If $\alpha > 2$: $I(u) = u \ln u + (1-u) \ln(1-u)$

$$* \quad f(\alpha, h) \stackrel{(1)}{=} \sup_{w \in [0,1]} \left(\frac{\alpha}{6} w^3 + \frac{h}{2} w - \frac{1}{2} I(w) \right) = \frac{\alpha}{6} w^*{}^3 + \frac{h}{2} w^* - \frac{1}{2} I(w^*)$$

where u^* denotes the/a maximizer of (1).

asymptotic approach

* } if u^* is unique: as $m \rightarrow \infty$, $P_{m,d,h}$ and P_{m,u^*}^{ER} have the same limit
 * } if u^* is not unique: as $m \rightarrow \infty$, $P_{m,d,h}$ behaves like $P_{m,u}^{ER}$ where u randomly chosen from the set of maximizers.

b. Phase diagram in (α, h) [Chatterjee, Dey '10 & Radin, Yim '13]

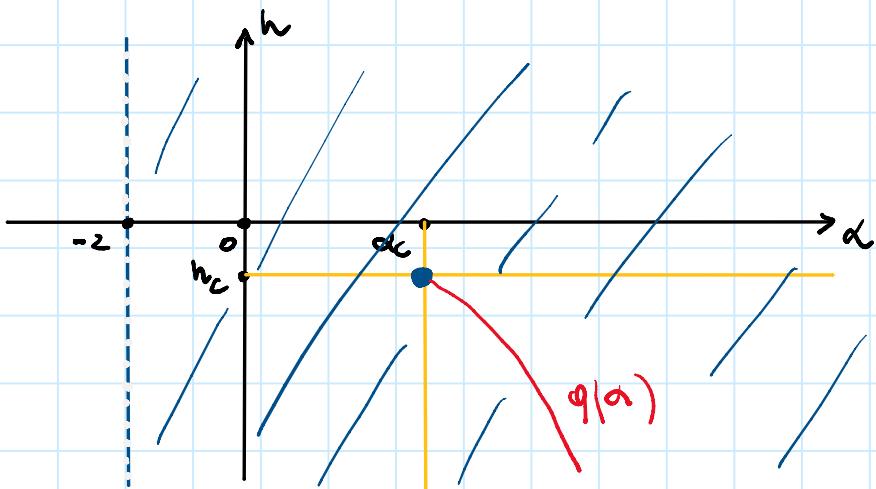
* $RS = \{(\alpha, h) : \alpha > -2\}$

Replica Symmetric regime

$$\rightarrow RS = \mathcal{U}^{RS} \cup \mathcal{M}^{RS}$$

where $\mathcal{U}^{RS} = \{(\alpha, h) : \alpha > -2, (1) \text{ has unique maximizer: } u^*\}$

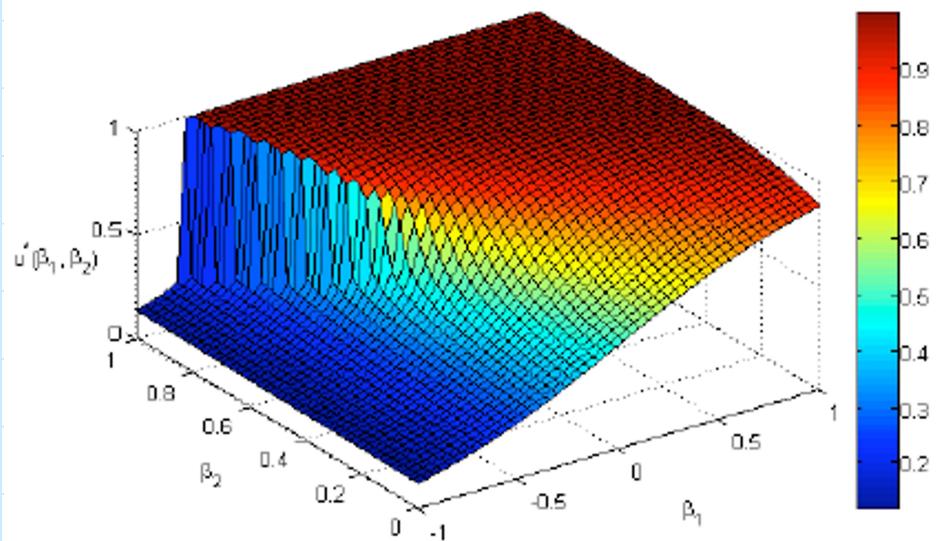
$\mathcal{M}^{RS} = \{(\alpha, h) : \alpha > -2, (1) \text{ has two maximizers: } u_n^* < u_2^*\}$



- $(\alpha_c, h_c) = \text{critical point} \in \mathcal{U}^{RS}$
 $\alpha_c = \frac{2\pi}{8}, h_c = \ln 2 - \frac{3}{2}$
- $q(\alpha) = \mathcal{M}^{RS}$

Thm [Radim, Yim '13]

- * If $(\alpha, h) \in \mathcal{U}^{\text{RS}} \setminus (\alpha_c, h_c)$ $\implies f(\alpha, h)$ is analytic
- * If $(\alpha, h) = (\alpha_c, h_c)$ $\implies f(\alpha_c, h_c)$ has second order discontinuity
- * If $(\alpha, h) \in \mathcal{M}^{\text{RS}}$ $\implies f(\alpha, h)$ has first order discontinuity



Consequences :

Non-uniqueness of $P_{m, \alpha, h}$
as $m \rightarrow \infty$ and $(\alpha, h) \in \mathcal{M}^{\text{RS}}$

(Picture by Suhade Fadnavis, taken
from [Chatterjee, Diaconis '13])

3. Main Results

a. Convergence of $\frac{2S_m}{m^2}$

Thm 1 [B., Collet, Maquinamie '21]

(i.) $\forall (\alpha, h) \in \mathcal{U}^{RS}$: $\frac{2S_m}{m^2} \xrightarrow[m \rightarrow \infty]{a.s.} u^*$ w.r.t. $P_{m; \alpha, h}$

(ii.) $\forall (\alpha, h) \in \mathcal{M}^{RS}$: $\frac{2S_m}{m^2} \xrightarrow[m \rightarrow \infty]{d} k \delta_{u_1^*} + (1-k) \delta_{u_2^*}$ w.r.t. $P_{m; \alpha, h}$

where $k \in (0, 1)$ is unknown.

Prop [Speed of convergence of SLN]

(a) $\exists \epsilon > 0$ $\forall \alpha \in (-2, \alpha_c) \Rightarrow \forall \epsilon > 0, \exists C_1, C_2 > 0 : \frac{C_1}{m} \leq \mathbb{E}_{m; \alpha, h} \left(\left| \frac{2S_m}{m^2} - u^* \right| \right) \leq \frac{C_2}{m^{1-\epsilon}}$

(b) $\forall (\alpha, h) = (\alpha_c, h_c) \Rightarrow \forall \epsilon > 0, \exists C_3, C_4 > 0 : \frac{C_3}{m^{1/2}} \leq \mathbb{E}_{m; \alpha_c, h_c} \left| \frac{2S_m}{m^2} - u^* \right| \leq \frac{C_4}{m^{1/2-\epsilon}}$

b. Fluctuations of $\frac{2S_m}{m^2}$

Set: $m_m \equiv m_m(\alpha, h) := E_{P_{m; \alpha, h}}\left(\frac{2S_m}{m^2}\right)$ average edge-density

$V_m \equiv V_m(\alpha, h) = \mathbb{D}_{P_{m; \alpha, h}}\left(\frac{2S_m}{m^2}\right)$ variance edge-density

Remark: $m_m = 2 \mathbb{D}_m f_m(\alpha, h) = \dots = E_m\left(\frac{2S_m}{m^2}\right)$

$V_m = 2 \mathbb{D}_m f_m(\alpha, h) = \dots = \frac{2}{m^2} \text{Var}(S_m) \geq 0$

Thm 2 [B., Collet, Maonamini '21]

$\forall (\alpha, h) \in \mathcal{U}^{RS} \setminus (\alpha_c, h_c)$, it holds the CLT

$$\frac{S_m - \frac{m^2}{2} m_m}{m/\sqrt{2}} \xrightarrow[m \rightarrow \infty]{d} N(0, V(\alpha, h)) \quad \text{w.r.t. } P_{m; \alpha, h}$$

where $V(\alpha, h) = \lim_{m \rightarrow \infty} V_m(\alpha, h) = 2 \mathbb{D}_m f(\alpha, h) = \mathbb{D}_m u^*$

c. Conjectures : [B., Bollet, Magnenuti '21]

Conjecture 1 : Non-standard limit theorem at (d_c, h_c)

$$\frac{S_m - \frac{m^2}{2} M_m}{m^{3/2}/2} \xrightarrow[m \rightarrow \infty]{d} Y \quad \text{w.r.t. } P_{m; d_c, h_c}$$

where Y is real r.v. with density $l_c(y) \propto e^{-\frac{81}{64}y^4}$

Conjecture 2 : Coefficients of the mixture of Dirac measures

$$\text{If } (\alpha, h) \in \mathcal{M}^{\text{rs}} : \frac{z S_m}{m^2} \xrightarrow[m \rightarrow \infty]{d} K \delta_{u_i^*} + (1-K) \delta_{u_2^*}$$

where $K = \frac{D(u_i^*)}{D(u_i^*) + D(u_2^*)}$ and $D(u_i^*) = (1 - 2\alpha(u_i^*)^2(1-u_i^*))^{-\frac{1}{2}}$

4. Mean-field approximation

a. Mean-field model: For $\underline{x} \in \{0,1\}^E_m$, let

$$\overline{P}_{m,\alpha,h}(\underline{x}) := e^{\overline{H}_{m,\alpha,h}(\underline{x})} / \overline{Z}_{m,\alpha,h}$$

where
$$\overline{H}_{m,\alpha,h}(\underline{x}) = \frac{\alpha}{3} \frac{1}{m^4} \left(\sum_{i \in E_m} x_i \right)^3 + h \sum_{i \in E_m} x_i = m^2 \left(\frac{\alpha}{6} u^3 + \frac{h}{2} u \right)$$

Hence:

replace $\frac{\alpha}{m} \sum_{\{i,j,k\} \in \binom{E}{3}} x_i x_j x_k$ $u = \frac{2S_m(\underline{x})}{m^2}$, with $u \in [0,1]_m$

$$\overline{Z}_{m,\alpha,h} = \sum_{u \in [0,1]_m} N_u e^{m^2 \left(\frac{\alpha}{6} u^3 + \frac{h}{2} u - \frac{1}{2} I(u) \right) + o(m^2)} = \sum_{u \in [0,1]_m} e^{\underbrace{m^2 \left(\frac{\alpha}{6} u^3 + \frac{h}{2} u - \frac{1}{2} I(u) \right)}_{U_{\alpha,h}(u)}}$$

Remark: $\sup_{u \in [0,1]} U_{\alpha,h}(u) = f(\alpha, h)$ → free energy of $P_{m,\alpha,h}$

b. Mean-field results [B., Collet, Magnanini '21]

Let $\bar{S}_n(\alpha, h) = \frac{1}{m^2} \lg \bar{\Sigma}_{m,\alpha,h}$ and $\bar{f}(\alpha, h) = \lim_{m \rightarrow \infty} \bar{S}_m(\alpha, h)$

Thm 3 [Free energy]

$$\forall (\alpha, h) \in RS; \quad \bar{f}(\alpha, h) = f(\alpha, h)$$

Thm 4 [Convergence of the edge density]

$$(i) \forall (\alpha, h) \in \mathcal{U}^{es}; \quad \frac{z S_m}{m^2} \xrightarrow[m \rightarrow \infty]{a.s.} u^*, \quad w.r.t. \bar{P}_{m,\alpha,h}$$

$$(ii) \forall (\alpha, h) \in \mathcal{M}^{RS}; \quad \frac{z S_m}{m^2} \xrightarrow[m \rightarrow \infty]{d} k S_{u_1^*} + (1-k) S_{u_2^*}, \quad w.r.t. \bar{P}_{m,\alpha,h}$$

where $k = \frac{D(u_1^*)}{D(u_1^*) + D(u_2^*)}$

Prop 2 : [Speed of convergence]

$$(i) \forall (\alpha, h) \in \mathcal{U}^{\text{es}} \setminus (\alpha_c, h_c): \lim_{n \rightarrow \infty} n \bar{E}_{m, \alpha, h} \left(\left| \frac{2S_m}{m^2} - u^* \right| \right) = C_1(\alpha, h) < \infty$$

$$(ii) \text{ At } (\alpha_c, h_c): \lim_{n \rightarrow \infty} n^{1/2} \cdot \bar{E}_{m, \alpha, h} \left(\left| \frac{2S_m}{m^2} - u^* \right| \right) = C_2(\alpha_c, h_c) < \infty$$

Thm 5 [Fluctuations of the edge density]

$$(i) \forall (\alpha, h) \in \mathcal{U}^{\text{rs}} \setminus (\alpha_c, h_c) \text{ and with } \bar{m}_m = \bar{E}_{m, \alpha, h} \left(\frac{2S_m}{m^2} \right)$$

$$\frac{S_m - \frac{m^2}{2} \bar{m}_m}{m/\sqrt{2}} \xrightarrow[m \rightarrow \infty]{d} N(0, \underline{\sigma^2(\alpha, h)}), \text{ w.r.t. } \bar{P}_{m, \alpha, h}$$

$\hookrightarrow 2\sigma^2_m f(\alpha, h)$

$$(ii) \text{ At } (\alpha_c, h_c): \boxed{\frac{S_m - \frac{m^2}{2} \bar{m}_m}{m^{3/2}/2} \xrightarrow[m \rightarrow \infty]{d} Y}, \text{ w.r.t. } \bar{P}_{m, \alpha_c, h_c}$$

where Y is real r.v. with density $l_c(y) \propto e^{-\frac{g_1}{64} y^4}$

5. Large deviations for edge-triangle model

Key tools

- (1) LDP for Erdős-Rényi graphs [Chatterjee, Varadhan '11]
- (2) LDP for integrals of exponential functional [Varadhan's Lemma]

$\xrightarrow{(1)+(2)}$ LDP for the edge-triangle model, with speed n^2 and rate function $J_{d,h}$, with minimizers provided in [Chatterjee, Diaconis '13].
 ↳ constant graphon \leftrightarrow constant connection probability among vertices = u^*

Heuristics on mom-standard CT at (α_c, h_c)

Setting $V_m = 2(S_m - \frac{m^2}{2} u^*) / m^{3/2}$:

$$P_{m,d,h_c}(V_m \in dx) = P_{m,d,h_c}\left(\frac{2S_m}{m^2} \in u^* + \frac{dx}{\sqrt{m}}\right) \approx e^{-m^2 J_{\alpha_c, h_c}(u^* + \frac{x}{m}) + o(m^2)} dx$$

↑ based on LDP

Taylor's expansion of $J_{d,h}$
up to forth-order

$$\approx \boxed{e^{-\frac{g_1}{64} x^4}} + o(m^2) dx$$

some density as mean-field model

Thanks for your
attention!!!