# Group Invariant States as Quantum Many-Body Scars

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Talk at CRM workshop

Conformal Field Theory and Quantum Many-Body Physics

CENTRE DE RECHERCHES MATHÉMATIQUES

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# Based Mainly On

- IRK, A. Milekhin, F. Popov, G. Tarnopolsky, "Spectra of Eigenstates in Fermionic Tensor Quantum Mechanics," PRD 97 (2018), 106023
- K. Pakrouski, IRK, F. Popov, G. Tarnopolsky, "Spectrum of Majorana Quantum Mechanics with $O(4)^3$ Symmetry," PRL 122 (2019), 011601
- K. Pakrouski, P. Pallegar, F. Popov, IRK, "Many Body Scars as a Group Invariant Sector of Hilbert Space," PRL 125 (2020), 230602.
- K. Pakrouski, P. Pallegar, F. Popov, IRK, "Group Theoretic Approach to Many Body Scar States in Fermionic Lattice Models," PRR 3 (2021), 043156
- Work in progress with K. Pakrouski, F. Popov, Z. Sun

# Singlet Sector Simplification

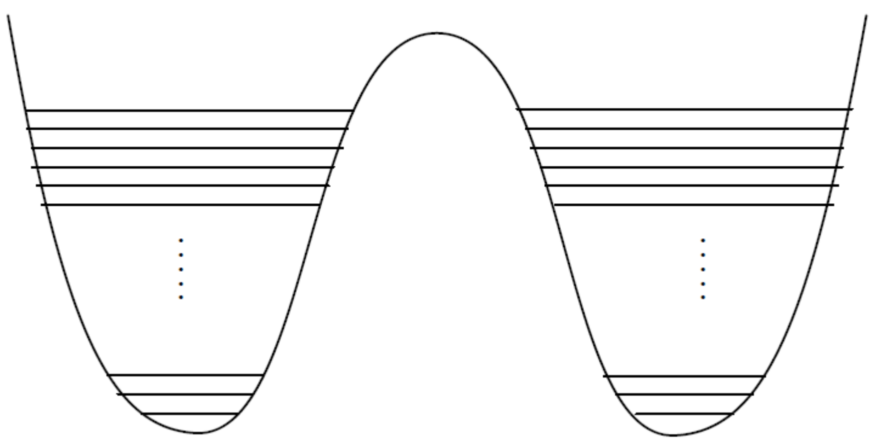
- Appears in many group invariant quantum many-body systems.
- For example, the SU(N) invariant sector of Hermitian matrix quantum mechanics is described by wave functions of N eigenvalues which act as the free fermions. Brezin et al.

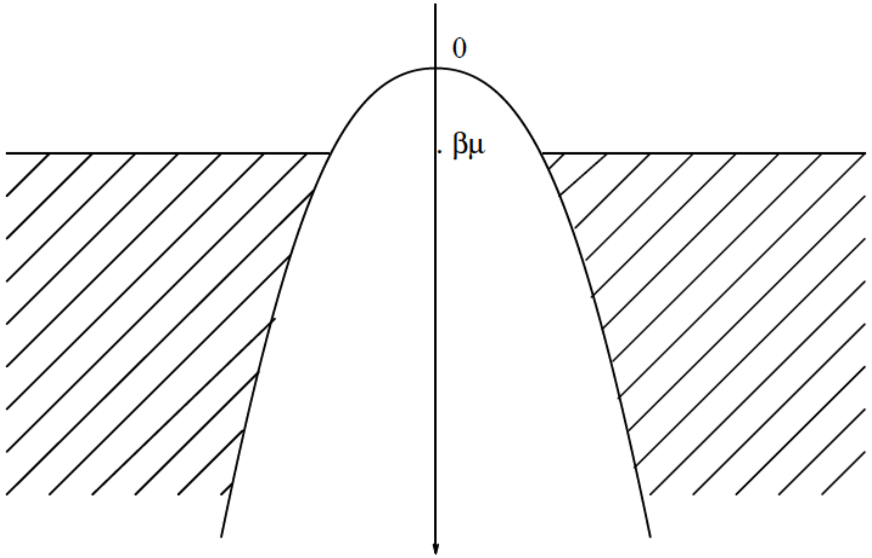
$$L = \mathrm{Tr}\{\tfrac{1}{2}\dot{\Phi}^2 - U(\Phi)\}$$

- The Vandermonde determinant $\prod_{i<j}(\lambda_i - \lambda_j)$ appears in $\Psi(\lambda) = \Delta(\lambda)\chi_{sym}(\lambda)$

$$\left(\sum_{i=1}^{N} h_i\right)\Psi(\lambda) = E\Psi(\lambda) \qquad h_i = -\frac{1}{2\beta^2}\frac{d^2}{d\lambda_i{}^2} + U(\lambda_i)$$

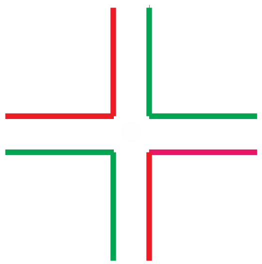
- The system of N free fermions can describe 2D string theory. Reviewed in hep-th/9108019
- The model with double-well potential, after double-scaling limit, describes 2D type 0B string theory. Takayanagi, Toumbas; Douglas, IRK, Kutasov, Maldacena, Martinec, Seiberg; Balthazar, Rodriguez, Yin; Sen; …





# O(N) x O(N) Matrix Model

- Theory of real matrices $\phi^{ab}$ with distinguishable indices, i.e. in the bi-fundamental representation of $O(N)_a x O(N)_b$ symmetry.
- The interaction is at least quartic: $g \operatorname{tr} \phi\phi^T\phi\phi^T$
- Propagators are represented by colored double lines, and the interaction vertex is



- In d=0 or 1 special limits describe two-dimensional quantum gravity.

# From Bi- to Tri-Fundamentals

- For a 3-tensor with distinguishable indices the propagator has index structure

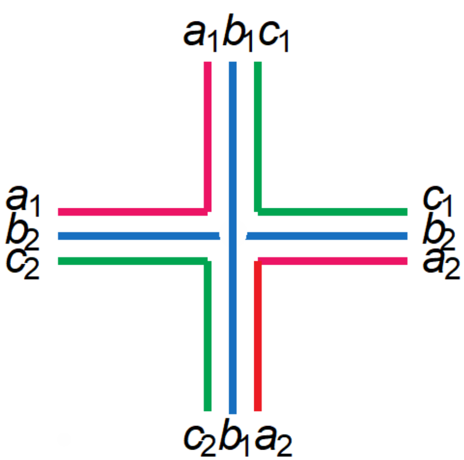
$$\langle \phi^{abc}\phi^{a'b'c'}\rangle = \delta^{aa'}\delta^{bb'}\delta^{cc'}$$

- It may be represented graphically by 3 colored wires 

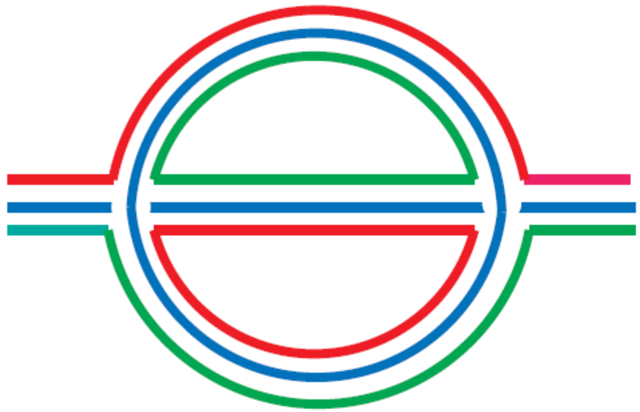

- Tetrahedral interaction with $O(N)_a \times O(N)_b \times O(N)_c$ symmetry

Carrozza, Tanasa; IK, Tarnopolsky

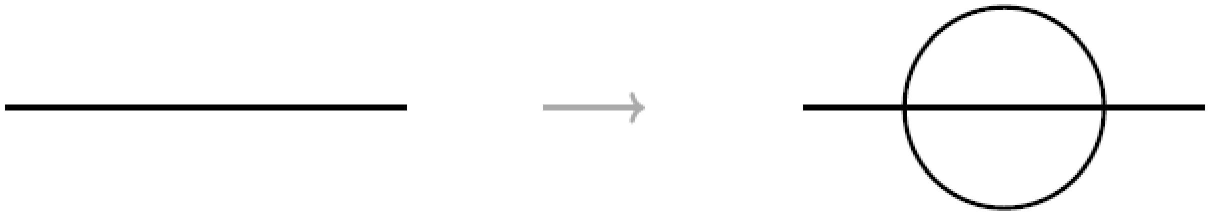
$$\frac{1}{4} g \phi^{a_1b_1c_1}\phi^{a_1b_2c_2}\phi^{a_2b_1c_2}\phi^{a_2b_2c_1}$$



- Leading correction to the propagator has 3 index loops



- Requiring that this "melon" insertion is of order 1 means that $\lambda = gN^{3/2}$ must be held fixed in the large N limit.
- Melonic graphs obtained by iterating



# $O(N)^3$ Tensor QM

- Quantum Mechanics of $N^3$ Majorana fermions
  IRK, Tarnopolsky

$$\{\psi^{abc}, \psi^{a'b'c'}\} = \delta^{aa'}\delta^{bb'}\delta^{cc'}$$

$$H = \frac{g}{4}\psi^{abc}\psi^{ab'c'}\psi^{a'bc'}\psi^{a'b'c} - \frac{g}{16}N^4$$

- Has $O(N)_a$x$O(N)_b$x$O(N)_c$ symmetry under

$$\psi^{abc} \to M_1^{aa'}M_2^{bb'}M_3^{cc'}\psi^{a'b'c'}, \qquad M_1, M_2, M_3 \in O(N)$$

- The SO(N) symmetry charges are

$$Q_1^{aa'} = \frac{i}{2}[\psi^{abc}, \psi^{a'bc}]\ , \qquad Q_2^{bb'} = \frac{i}{2}[\psi^{abc}, \psi^{ab'c}]\ , \qquad Q_3^{cc'} = \frac{i}{2}[\psi^{abc}, \psi^{abc'}]$$

# $O(N)^3$ vs. SYK Model

- Using composite indices $I_k = (a_k b_k c_k)$

$$H = \frac{1}{4!} J_{I_1 I_2 I_3 I_4} \psi^{I_1} \psi^{I_2} \psi^{I_3} \psi^{I_4}$$

The couplings take values $0, \pm 1$

$$J_{I_1 I_2 I_3 I_4} = \delta_{a_1 a_2} \delta_{a_3 a_4} \delta_{b_1 b_3} \delta_{b_2 b_4} \delta_{c_1 c_4} \delta_{c_2 c_3} - \delta_{a_1 a_2} \delta_{a_3 a_4} \delta_{b_2 b_3} \delta_{b_1 b_4} \delta_{c_2 c_4} \delta_{c_1 c_3} + 22 \text{ terms}$$

- The number of distinct terms is

$$\frac{1}{4!} \sum_{\{I_k\}} J^2_{I_1 I_2 I_3 I_4} = \frac{1}{4} N^3 (N-1)^2 (N+2)$$

- Much smaller than in SYK model with $N_{\mathrm{SYK}} = N^3$

$$\frac{1}{24} N^3 (N^3 - 1)(N^3 - 2)(N^3 - 3)$$

# Gauged Model

- To eliminate large degeneracies, gauge the symmetry. Witten
- Focus on the states invariant under $SO(N)^3$.
- Their number can be found by gauging the free theory IRK, Milekhin, Popov, Tarnopolsky

$$\#\text{singlet states} = \int d\lambda_G^N \prod_{a=1}^{M/2} 2\cos(\lambda_a/2)$$

$$d\lambda_{SO(2n)} = \prod_{i<j}^{n} \sin\left(\frac{x_i - x_j}{2}\right)^2 \sin\left(\frac{x_i + x_j}{2}\right)^2 dx_1 \dots dx_n$$

- No singlets for odd N due to a QM anomaly.
- The number grows very rapidly for even N

| $N$ | # singlet states |
|---|---|
| 2 | 2 |
| 4 | 36 |
| 6 | 595354780 |

Table 1: Number of singlet states in the $O(N)^3$ model

$$\#\text{singlet states} \sim \exp\left(\frac{N^3}{2}\log 2 - \frac{3N^2}{2}\log N + O(N^2)\right)$$

- Large N dynamics in the singlet sector is similar to SYK. Same melonic Schwinger-Dyson eqns.
- The large low-temperature entropy suggests tiny gaps for singlet excitations ~ $c^{-N^3}$

# Qubit Hamiltonian

- Convenient to introduce operator basis which breaks the third O(N) to U(N/2)

$$\bar{c}_{abk} = \frac{1}{\sqrt{2}}\left(\psi^{ab(2k)} + i\psi^{ab(2k+1)}\right), \quad c_{abk} = \frac{1}{\sqrt{2}}\left(\psi^{ab(2k)} - i\psi^{ab(2k+1)}\right),$$

$$\{c_{abk}, c_{a'b'k'}\} = \{\bar{c}_{abk}, \bar{c}_{a'b'k'}\} = 0, \quad \{\bar{c}_{abk}, c_{a'b'k'}\} = \delta_{aa'}\delta_{bb'}\delta_{kk'},$$

$$a, b = 0, 1, \ldots, N-1, \text{ and } k = 0, \ldots, \tfrac{1}{2}N - 1$$

- Operators $c_{abk}, \bar{c}_{abk}$ correspond to qubit number $N^2k + Nb + a$
- The Hamiltonian couples N/2 sets of $N^2$ qubits

$$H = 2\left(\bar{c}_{abk}\bar{c}_{ab'k'}c_{a'bk'}c_{a'b'k} - \bar{c}_{abk}\bar{c}_{a'bk'}c_{ab'k'}c_{a'b'k}\right)$$

- The Cartan generators of U(N/2) are

$$Q_k = \sum_{a,b} \frac{1}{2}[\bar{c}_{abk}, c_{abk}] \ , \qquad k = 0, \ldots, \frac{1}{2}N - 1$$

- For the oscillator vaccuum

$$c_{abk} \left|\mathrm{vac}\right\rangle = 0 \ , \qquad Q_k \left|\mathrm{vac}\right\rangle = -\frac{N^2}{2} \left|\mathrm{vac}\right\rangle$$

- The gauge singlet states appear in the sector where all these charges vanish: each set of $N^2$ qubits is at half filling.
- This reduces the number of states but it still grows rapidly. For N=4 there are 165636900, while for N=6 over 7.47 * 10^29

# Spectrum of the Gauged N=4 Model

- Studied the system of 32=16+16 qubits

  K. Pakrouski, IK, F. Popov and G. Tarnopolsky

- Needed to isolate the 36 states invariant under $SO(4)^3$ out of the 165080390 “half-half-filled” states.
- Diagonalize 4H/g + 100 C where C is the sum of three Casimir operators.
- A Lanczos type algorithm is well suited for this sparse operator.
- Find 15 distinct $SO(4)^3$ invariant energy levels: E=0 and 7 “mirror pairs” (E, -E).

# Discrete Symmetries

- Act within the SO(N)$^3$ invariant sector and can lead to small degeneracies.
- $Z_2$ parity transformation within each group like

$$\psi^{1bc} \to -\psi^{1bc}$$

- Interchanges of the groups flip the energy

$$P_{23}\psi^{abc}P_{23} = \psi^{acb} \ , \qquad P_{12}\psi^{abc}P_{12} = \psi^{bac}$$

$$P_{23}HP_{23} = -H \ , \qquad P_{12}HP_{12} = -H$$

- $Z_3$ symmetry generated by $P = P_{12}P_{23}$ , $\qquad P^3 = 1$

$$P\psi^{abc}P^\dagger = \psi^{cab} \ , \qquad PHP^\dagger = H$$

- At non-zero energy the gauge singlet states transform under the discrete group $A_4 \times Z_2$.
- Spectrum for N=4. Pakrouski, IK, Popov, Tarnopolsky

$$\pm\sqrt{32(447 \pm \sqrt{125601})}$$

$$\pm\sqrt{32(187 \pm \sqrt{11481})}$$

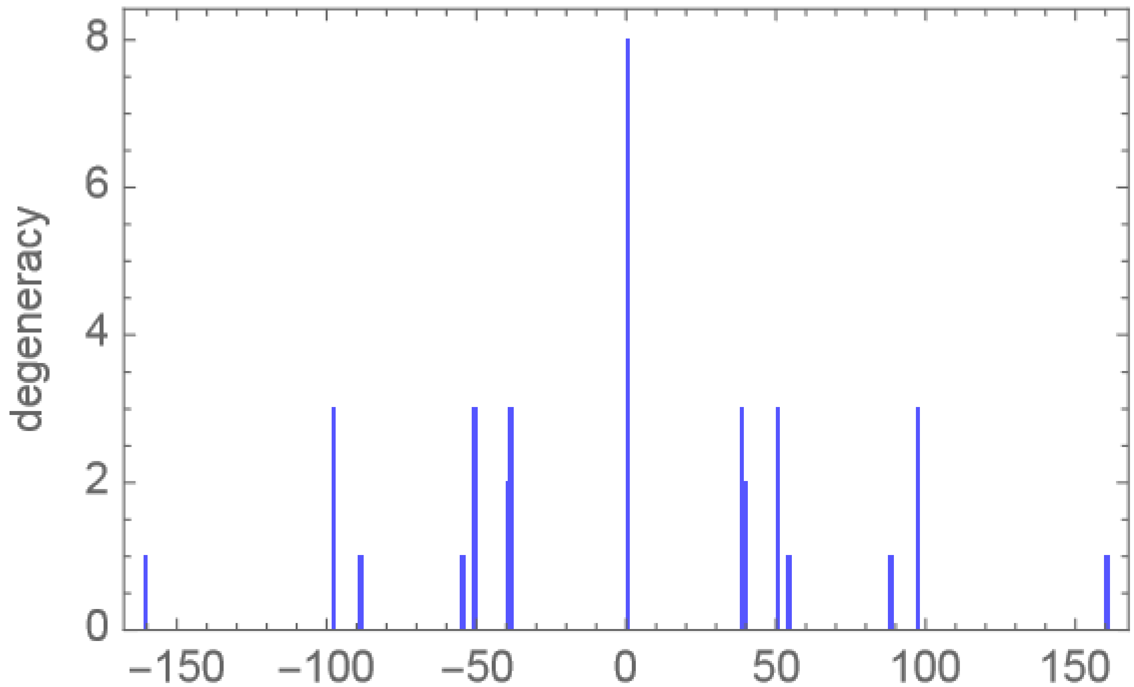
$$8\sqrt{24} =$$

$$8\sqrt{23} =$$

| $E$ | $P_1$ | $P_2$ | $P_3$ | $E$ | $P_1$ | $P_2$ | $P_3$ |
|---|---|---|---|---|---|---|---|
| −160.140170 | 1 | 1 | 1 | 160.140170 | 1 | 1 | 1 |
| −97.019491 | 1 | 1 | −1 | 97.019491 | 1 | 1 | −1 |
| −97.019491 | −1 | 1 | 1 | 97.019491 | −1 | 1 | 1 |
| −97.019491 | 1 | −1 | 1 | 97.019491 | 1 | −1 | 1 |
| −88.724292 | −1 | −1 | −1 | 88.724292 | −1 | −1 | −1 |
| −54.434603 | 1 | 1 | 1 | 54.434603 | 1 | 1 | 1 |
| −50.549167 | 1 | 1 | −1 | 50.549167 | 1 | 1 | −1 |
| −50.549167 | −1 | 1 | 1 | 50.549167 | −1 | 1 | 1 |
| −50.549167 | 1 | −1 | 1 | 50.549167 | 1 | −1 | 1 |
| −39.191836 | 1 | 1 | 1 | 39.191836 | 1 | 1 | 1 |
| −39.191836 | 1 | 1 | 1 | 39.191836 | 1 | 1 | 1 |
| −38.366652 | 1 | −1 | −1 | 38.366652 | 1 | −1 | −1 |
| −38.366652 | −1 | 1 | −1 | 38.366652 | −1 | 1 | −1 |
| −38.366652 | −1 | −1 | 1 | 38.366652 | −1 | −1 | 1 |
| 0.000000 | 1 | 1 | 1 | 0.000000 | −1 | −1 | −1 |
| 0.000000 | −1 | 1 | 1 | 0.000000 | 1 | −1 | −1 |
| 0.000000 | 1 | −1 | 1 | 0.000000 | −1 | 1 | −1 |
| 0.000000 | 1 | 1 | −1 | 0.000000 | −1 | −1 | 1 |

**arXiv:1808.07455**

# Singlet Energies for N=4




- For N=6 there will be over 595 million states packed into energy interval <1932. So, the gaps should be tiny. Pakrouski, IRK, Popov, Tarnopolsky

# Unequal Ranks

- Generalize the Majorana tensor model to have $O(N_1) \times O(N_2) \times O(N_3)$ symmetry
- The traceless Hamiltonian is

$$H = \frac{g}{4}\psi^{abc}\psi^{ab'c'}\psi^{a'bc'}\psi^{a'b'c} - \frac{g}{16}N_1N_2N_3(N_1 - N_2 + N_3)$$

$$\{\psi^{abc}, \psi^{a'b'c'}\} = \delta^{aa'}\delta^{bb'}\delta^{cc'}$$

$$a = 1, \ldots, N_1;\ b = 1, \ldots, N_2;\ c = 1, \ldots, N_3$$

- The Hilbert space has dimension $2^{[N_1N_2N_3/2]}$
- The eigenstates of H form irreducible representations of the symmetry.

# Energy Bounds

- The bound on the singlet ground state energy
  IRK, Milekhin, Popov, Tarnopolsky

$$|E| \le E_{bound} = \frac{g}{16} N^3 (N+2)\sqrt{N-1}$$

- In the melonic limit, this correctly scales as $N^3$.
- The gap to the lowest non-singlet state scales as 1/N.
- For unequal ranks the bound is

$$|E| \le \frac{g}{16} N_1 N_2 N_3 (N_1 N_2 N_3 + N_1^2 + N_2^2 + N_3^2 - 4)^{1/2}$$

# A Fermionic Matrix Model

- For $N_3$=2 the bound simplifies to

$$|E|_{N_3=2} \leq \frac{g}{8} N_1 N_2 (N_1 + N_2)$$

- Saturated by the ground state.
- This is a fermionic matrix model with symmetry

$$O(N_1) \times O(N_2) \times U(1)$$

$$\bar{\psi}_{ab} = \frac{1}{\sqrt{2}} \left( \psi^{ab1} + i\psi^{ab2} \right), \quad \psi_{ab} = \frac{1}{\sqrt{2}} \left( \psi^{ab1} - i\psi^{ab2} \right)$$

$$\{\bar{\psi}_{ab}, \bar{\psi}_{a'b'}\} = \{\psi_{ab}, \psi_{a'b'}\} = 0, \quad \{\bar{\psi}_{ab}, \psi_{a'b'}\} = \delta_{aa'}\delta_{bb'}$$

- The traceless Hamiltonian is

$$H = \frac{g}{2}\left(\bar{\psi}_{ab}\bar{\psi}_{ab'}\psi_{a'b}\psi_{a'b'} - \bar{\psi}_{ab}\bar{\psi}_{a'b}\psi_{ab'}\psi_{a'b'}\right) + \frac{g}{8}N_1 N_2 (N_2 - N_1)$$

- Describes qubits on $N_1$ x $N_2$ lattice with non-local couplings.
- May be expressed in terms of quadratic Casimirs

$$-\frac{g}{2}\left(4C_2^{SU(N_1)} - C_2^{SO(N_1)} + C_2^{SO(N_2)} + \frac{2}{N_1}Q^2 + (N_2 - N_1)Q - \frac{1}{4}N_1 N_2 (N_1 + N_2)\right)$$

- $SU(N_1) \times SU(N_2)$ is not a symmetry here but an enveloping algebra. Some states have enhanced symmetry: they are SU(N) invariant.

# Complete Spectrum

- The singlets “scar” the energy distribution.

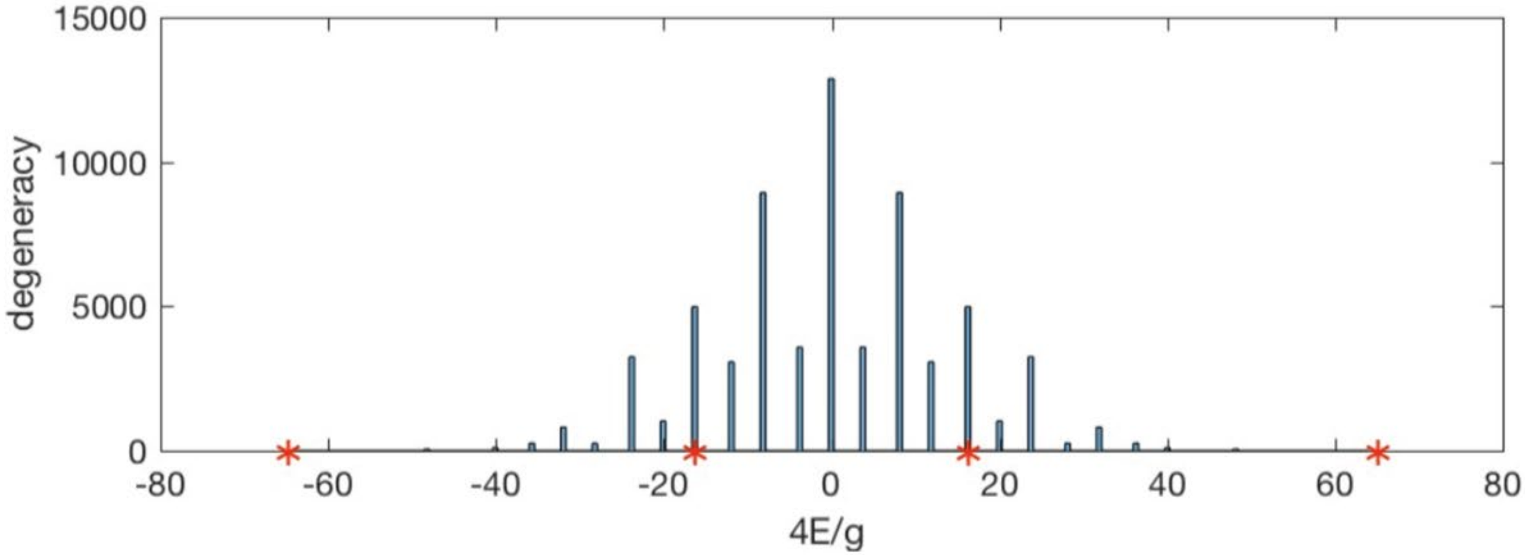



Figure 1: Spectrum of the $O(4)^2 \times O(2)$ model. There are four singlet states, and the stars mark their energies.

# Towards Hubbard Model

- Can also think of the first index as labeling the lattice site, and the second as labeling spin. When $N_2$=2, there are two spin states, up and down. The model is beginning to resemble a non-local Hubbard model, but need to add quadratic hopping terms. Pakrouski, Pallegar, Popov, IRK
- Imaginary hopping terms are SO(N) generators

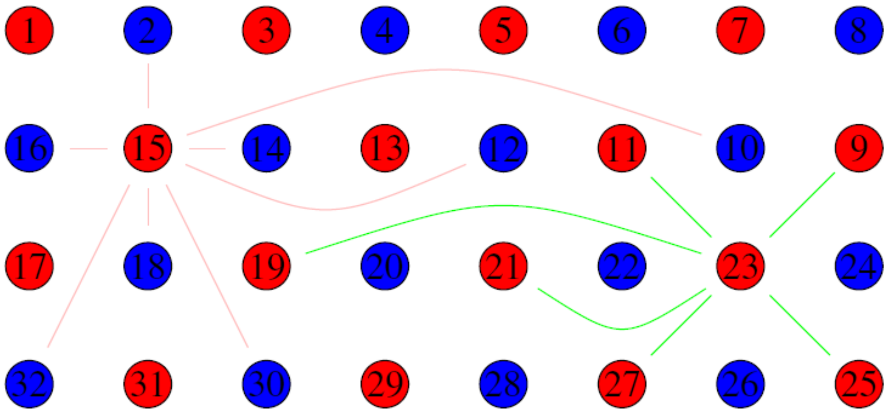
$$T^{O}_{kl} = i \sum_{\sigma} (c^{\dagger}_{k\sigma} c_{l\sigma} - c^{\dagger}_{l\sigma} c_{k\sigma}) \qquad \sigma = \uparrow, \downarrow$$

- Adding them to H keeps singlets as eigenstates but mixes up the non-singlets.

- A simple transformation leads to a model with a real nearest neighbor hopping parameter:

$$H_{nn} = t \sum_{\langle ij \rangle \sigma} (c^{\dagger}_{i\sigma} c_{j\sigma} + h.c.)$$

- This transformation is possible on a bi-partite lattice



# SO(N) Singlets as Scars

- For the eigenstates that are SO(N) singlets, the energy is independent of the hopping parameter $t$. Pakrouski, Pallegar, Popov, IRK
- This is the kind of simplification characteristic of the Quantum Many-Body Scars, that have been an active area in Condensed Matter Physics. Reviews by Serbyn, Abanin, Papic; Moudgalya, Bernevig, Regnault; Chandran, Iadecola, Khemani, Moessner
- Scars form an "integrable subsector" in a Hamiltonian that is in general not integrable.

# Group Theoretic Approach to Scars

- There are Hamiltonians that are not symmetric under a Lie group G, yet some of their eigenstates are invariant. These are the quantum many-body scars!
- A class of such Hamiltonians is

$$H = H_0 + \sum_j O_j T_j^G$$

- Includes local lattice systems like the tJU model

$$H^{tJU} = \sum_{\langle ij \rangle \sigma} (t c_{i\sigma}^{\dagger} c_{j\sigma} + h.c.) + J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j + U \sum_i n_{i\uparrow} n_{i\downarrow} - \mu Q$$

# Pseudospin

- An example is provided by C.N. Yang's eta-pairing states in the Hubbard model on a bipartite lattice.
- After transforming to imaginary hopping, the pseudospin group $\widetilde{\mathrm{SU}}(2)$ is generated by

$$\eta^+ = \sum_j c^\dagger_{j\uparrow} c^\dagger_{j\downarrow} = \frac{1}{2} \sum_{j,\sigma,\sigma'} c^\dagger_{j\sigma} c^\dagger_{j\sigma'} \epsilon_{\sigma\sigma'}$$

$$\eta^- = (\eta^+)^\dagger, \quad \eta^3 = \frac{1}{2}(Q - N) \qquad Q = \sum_{i=1}^{N} n_i$$

$$n_{i\uparrow} = c^\dagger_{i\uparrow} c_{i\uparrow} \,, \quad n_{i\downarrow} = c^\dagger_{i\downarrow} c_{i\downarrow} \,, \quad n_i = n_{i\uparrow} + n_{i\downarrow}$$

- It commutes with the rotation group $\mathrm{SU}(2)$ and with the O(N) that acts on the lattice index.

# Eta-pairing states

- There are N+1 states that are SU(2) invariant and form a multiplet of pseudospin N/2 Yang, Zhang

$$|n^\eta\rangle = \frac{(\eta)^n}{\sqrt{\frac{N!n!}{(N-n)!}}} |0\rangle \ , \qquad n = 0, \ldots, N$$

- The fact that they are also O(N) invariant was pointed out recently. Pakrouski et al.
- In fact, they are invariant under a bigger group $\widetilde{\mathrm{Sp}}(N)$
- They are highly excited, equally spaced states that play the role of scars in the (deformed) Hubbard model. Moudgalya et al.; Mark, Motrunich

# Groups Acting on Hilbert Space

- So far, we have used $\mathrm{SU}(2) \times \widetilde{\mathrm{SU}}(2) \times \mathrm{O}(N)$ but there are bigger groups of which this is a subgroup: $\mathrm{SU}(2) \times \mathrm{Sp}(N) \supset \mathrm{SU}(2) \times \mathrm{U}(N)$

$$\widetilde{\mathrm{SU}}(2) \times \widetilde{\mathrm{Sp}}(N) \supset \widetilde{\mathrm{SU}}(2) \times \widetilde{\mathrm{U}}(N)$$

$$\mathrm{U}(N) : E_{AB} = c^\dagger_{A\uparrow} c_{B\uparrow} - c_{B\downarrow} c^\dagger_{A\downarrow} = c^\dagger_{A\uparrow} c_{B\uparrow} + c^\dagger_{A\downarrow} c_{B\downarrow} - \delta_{AB}$$

$$\widetilde{\mathrm{U}}(N) : \widetilde{E}_{AB} = c^\dagger_{A\uparrow} c_{B\uparrow} - c^\dagger_{B\downarrow} c_{A\downarrow} = c^\dagger_{A\uparrow} c_{B\uparrow} + c_{A\downarrow} c^\dagger_{B\downarrow} - \delta_{AB}$$

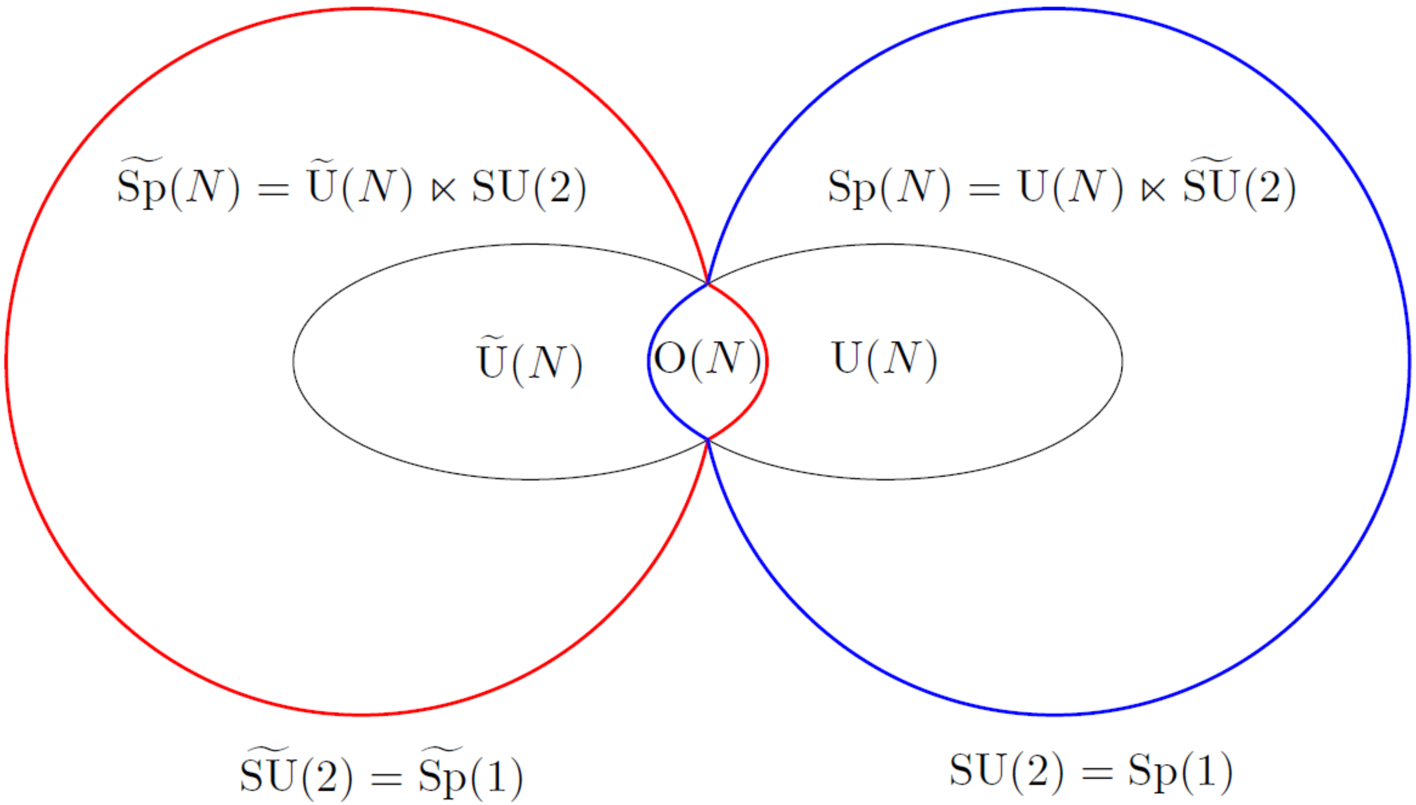
- Generators of $\widetilde{\mathrm{Sp}}(N)$ are

$$\widetilde{F}^0_{AB} = E_{A\uparrow,B\uparrow} - E_{B\downarrow,A\downarrow} = c^\dagger_{A\uparrow} c_{B\uparrow} - c^\dagger_{B\downarrow} c_{A\downarrow}$$

$$\widetilde{F}^+_{(AB)} = E_{A\uparrow,B\downarrow} + E_{B\uparrow,A\downarrow} = c^\dagger_{A\uparrow} c_{B\downarrow} + c^\dagger_{B\uparrow} c_{A\downarrow}$$

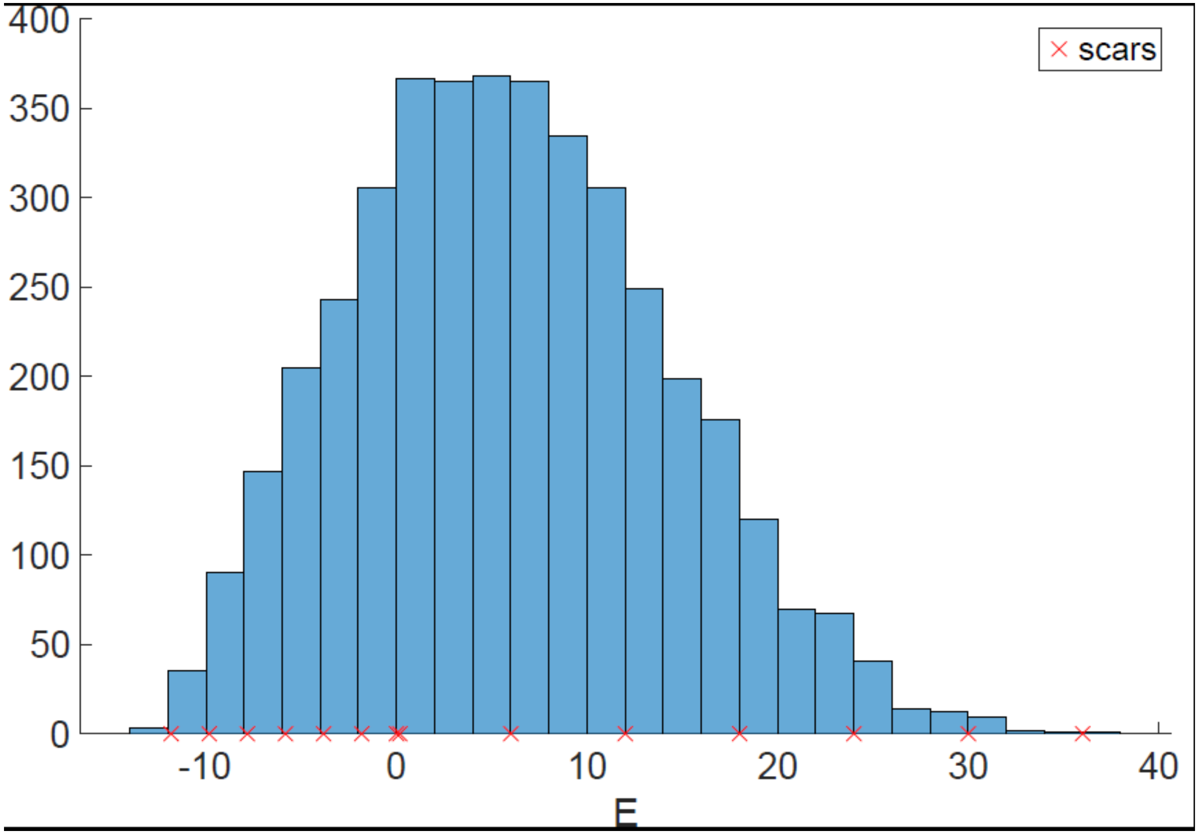
$$\widetilde{F}^-_{(AB)} = E_{A\downarrow,B\uparrow} + E_{B\downarrow,A\uparrow} = c^\dagger_{A\downarrow} c_{B\uparrow} + c^\dagger_{B\downarrow} c_{A\uparrow}$$

- Schematic structure of the groups acting on the Hilbert space for a lattice of N sites with a spin-1/2 fermion on each site



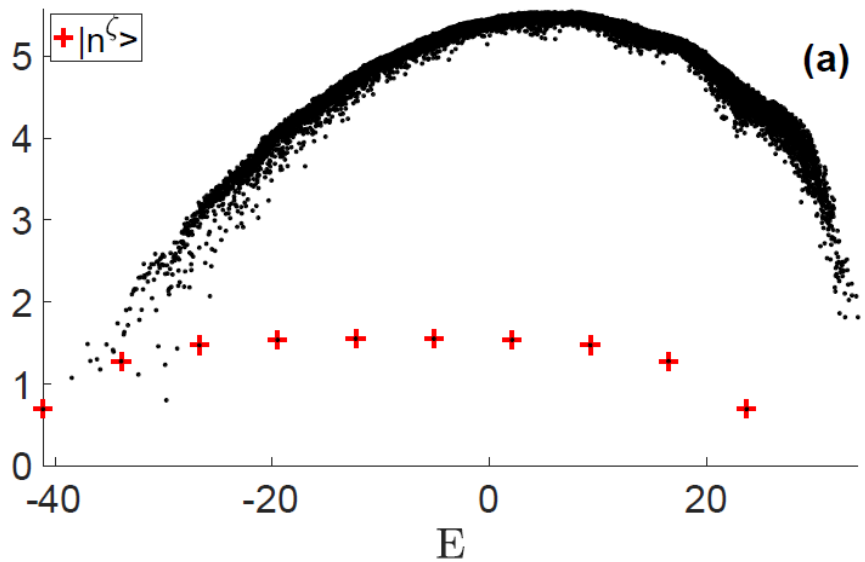
# Histogram of the Energy Levels

- A typical distribution for the tJU model deformed by OT terms



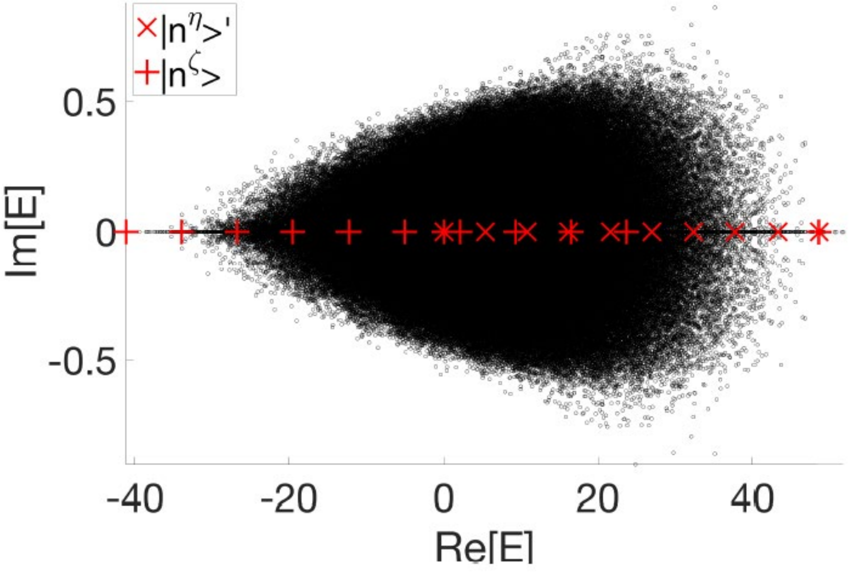
# Low Entanglement

- The scar states are distinguished by their low entanglement entropy when the system is divided into two parts:



# Non-Hermitian Hamiltonians

- The group theoretic approach to scars continues to work when non-Hermitian terms are added to the Hamiltonians, e.g. the tJU model.
- The energies of scars continue to be real



# Comments

- The scar states, which are invariant under the large Lie group acting on the lattice sites, are decoupled from all the non-singlet states. Only the latter thermalize.
- This decoupling survives the OT perturbations and may approximately survive some other perturbations.
- The Group theoretic approach to scars applies to non-Hermitian Hamiltonians.
- Scar states in QFT? In AdS/CFT?