Box-Ball Systems from Integrable Systems and Probabilistic Perspectives

CRM Workshop, 19-23 September 2022

Centre de Recherches Mathématiques, Université de Montréal

Organisers: David Croydon (Kyoto University), Makiko Sasada (University of Tokyo), Satoshi Tsujimoto (Kyoto University)

Monday 19th September	Tuesday 20th September	Wednesday 21st September	Thursday 22nd September	Friday 23rd September
10:00–10:50 Takahashi*	10:00–10:50 Tsujimoto	10:00–10:50 Tsujimoto	10:00–10:50 Kuniba*	10:00–10:50 Kuniba*
11:00–11:50 Ferrari*	11:00–11:50 Sasada	11:00–11:50 Ferrari*	11:00–11:50 Sasada	11:00–11:50 Croydon
13:30–14:20 Okado*	13:30–14:20 O'Connell*		13:30–14:20 Willox	13:00–13:50 Doyon
14:30–15:20 Suda*	14:30–15:20 Quastel*	Free discussion	14:30–15:20 Lyu	14:00–14:50 Spohn*
15:40–16:30 Maeda	15:40–16:30 Sasamoto*		15:40–16:30 Wang*	15:10–16:00 Pasquier
	16:40–17:30 Corwin		16:40–17:30 Franceschini	

Coffee/pastries 9:30–10:00 each morning. Coffee after second afternoon talk (not Wednesday). $*$ = online participant.

Morning lectures

Daisuke Takahashi* ..... *Viewpoints brought by the box and ball system*
Pablo Ferrari* ..... *Slot decomposition of the box-ball system*
Satoshi Tsujimoto ..... *Discrete integrable systems and their ultradiscretizations*
Makiko Sasada ..... *Bi-infinite dynamics and stationary measures for discrete KdV- and Toda-type systems*
Atsuo Kuniba* ..... *Integrability of box-ball systems and randomized box-ball systems*
David Croydon ..... *Generalized hydrodynamic limit for the box-ball system*

Afternoon talks

Masato Okado* ..... *Box-ball systems: reviews and recent developments*
Hayate Suda* *Relationships between two linearizations of the box-ball system: rigged configuration and slot decomposition*
Kazuki Maeda ..... *Box-ball systems and biorthogonal polynomials*
Neil O'Connell* ..... *Toda and probability*
Jeremy Quastel* ..... *Polynuclear growth and the Toda lattice*
Tomohiro Sasamoto* ..... *BBS like behaviors in skew RSK dynamics*
Ivan Corwin ..... *Invariant measures for KPZ systems with boundaries*
Ralph Willox ..... *Direct and inverse scattering for the ultradiscrete KdV equation*
Hanbaek Lyu ..... *Scaling limit of soliton statistics of a multicolor box-ball system*
Minmin Wang* ..... *Soliton speeds in the box-ball system*
Chiara Franceschini ..... *Some results for hard rods of inhomogeneous size*
Benjamin Doyon ..... *Generalised hydrodynamics for classical particle systems*
Herbert Spohn* ..... *Spacetime correlations of the classical Toda chain in thermal equilibrium*
Vincent Pasquier ..... *Hydrodynamics of a box-ball system*

`http://www.crm.umontreal.ca/2022/BBS22/index_e.php`

Morning lectures

Daisuke Takahashi* (Waseda University)

Viewpoints brought by the box and ball system

The box and ball system is an ultimately discretized integrable system, realizing solitons in a digital system. It was first discovered by a cellular automaton as an advection system. It can be obtained by ultradiscretization (= tropicalization, crystallization) which gives a max-plus system as a kind of limit of a difference soliton equation and a cellular automaton is embedded in the max-plus system.

In this talk, I will briefly introduce the discovery of the box and ball system and its subsequent progress, and then describe the expansion of research based on various viewpoints, including traffic flow, pattern formation, chaos, Lyapunov functions, and lattice equations with a focus on ultradiscretization, max-plus and max-min representations.

Pablo Ferrari* (Universidad de Buenos Aires)

Slot decomposition of the box-ball system

The box-ball system is a transport cellular automaton of balls on the integers. In 1990 Takahashi and Satsuma observed that the dynamics conserves solitons, solitary waves travelling at speed proportional to its size, that conserve shape and speed even after colliding with other solitons. We introduce the concept of "slot" to describe a ball configuration via its "soliton components". There is a component for each size k soliton, k a natural number. The components evolve as a hierarchical shift, where the amount each component is shifted depends on the bigger components. A random initial ball configuration with translation invariant and independent components is invariant for the dynamics. The soliton decomposition of an iid Bernoulli configuration with density less than $1/2$ has independent components, and for each k, the k-th component consists of iid geometric random variables, with parameter depending on k. The slot decomposition of an excursion of the random walk with increments given by the ball configuration gives an alternative way to construct the Harris correspondence between the excursion and a tree.

The talks are based on joint works with Chi Nguyen, Leo Rolla, Minmin Wang and Davide Gabrielli.

Satoshi Tsujimoto (Kyoto University)

Discrete integrable systems and their ultradiscretizations

As a review of discrete integrable systems related to box-ball systems, procedures for deriving box-ball systems from integrable discrete systems such as the KdV equation known as the soliton equation and Toda lattices on finite lattices will be introduced. In the course of the lecture, conserved quantities, exact solutions, and some extensions of the box-ball system will be discussed, as well as the relationship between the box-ball system and invariant factors of integer matrices among recent studies.

Makiko Sasada (University of Tokyo)

Bi-infinite dynamics and stationary measures for discrete KdV- and Toda-type systems

1. Bi-infinite dynamics for KdV- and Toda-type discrete integrable systems based on generalized Pitman's transformation

In this talk, we introduce bi-infinite versions of the box-ball system (BBS) and four well-studied discrete integrable models, namely the ultra-discrete KdV equation, the discrete KdV equation, the ultra-discrete Toda lattice, and the discrete Toda lattice. The BBS is known to be a special case of the ultra-discrete KdV equation as well as the ultra-discrete Toda lattice. For each dynamics, we show that there exists a unique solution to the initial value problem when the given initial data lies within a certain class, which includes the support of many shift ergodic measures. Our unified approach, which is also applicable to other integrable systems defined locally via lattice maps, involves the introduction of a path encoding (that is, a certain antiderivative) of the model configuration, and the dynamic is characterized by generalizations of the classical 'Pitman's transformation' of reflection in the past maximum, which is well-known to probabilists.

2. Stationary measures for discrete KdV- and Toda-type systems

In this talk, we study stationary measures of the box-ball system (BBS) and four well-studied discrete integrable models, namely the ultra-discrete KdV equation, the discrete KdV equation, the ultra-discrete Toda lattice, and the discrete Toda lattice. We give a unified approach to obtain a sufficient condition for stationarity and also characterize spatially independent and identical/alternate stationary distributions. For the BBS, we also give some explicit spatially non i.i.d. stationary measures including periodic Generalized Gibbs ensembles.

Atsuo Kuniba* (University of Tokyo)

Integrability of box-ball systems and randomized box-ball systems

1. Integrability of box-ball systems: crystals, Bethe ansatz and ultradiscretization

BBS emerges from classical integrable systems via ultradisretation and also as quantum integrable systems at $q = 0$. I shall survey the fruitful synthesis of the double origin of the integrability that took place in the last two decades. Main ingredients are combinatorial Bethe ansatz and crystals theory in quantum groups. They acquire natural interpretation in soliton theory, e.g, rigged configurations as action-angle variables, KKR bijection as inverse scattering transformation, fermionic character formula as partition function, corner transfer matrices as tau functions, Bethe eigenvalues at $q = 0$ as Poincare cycles, etc. If time permits, I shall describe the recently formulated complete box ball system in the higher rank which has a completely diagonal S matrix.

2. Randomized box-ball systems: density plateaux, current correlations and large deviations

I shall report analytical and simulation results on randomized BBS obtained recently by TBA and GHD. They include, exact solution of TBA equation via Q and Y-systems, limit shape of conserved Young diagrams, solution of GHD speed equation, density plateaux generated from domain wall initial conditions including their diffusive broadening, long time behavior of correlations of generalized currents extending Drude weight, large deviation function for ball currents, etc. The contents are based on joint works with H. Lyu, M. Okado and G. Misguich, V. Pasquier.

David Croydon (Kyoto University)

Generalized hydrodynamic limit for the box-ball system

I will present a generalized hydrodynamic limit for the box-ball system, which explains how the densities of solitons of different sizes evolve asymptotically under Euler space-time scaling. The limiting soliton flow is described in terms of a continuous state-space analogue of the soliton decomposition of Ferrari, Nguyen, Rolla and Wang (cf. the original work of Takahashi and Satsuma), namely the densities of solitons of given sizes in space are related to corresponding densities on a scale of 'effective distances', where the dynamics are linear. For smooth initial conditions, I will further explain how the resulting evolution of the soliton densities in space can alternatively be characterised by a partial differential equation, which naturally links the time-derivatives of the soliton densities and the 'effective speeds' of solitons locally. This is joint work with Makiko Sasada (University of Tokyo).

Afternoon talks

Masato Okado* (Osaka Metropolitan University)

Box-ball systems: reviews and recent developments

I review why the box-ball system can be considered as integrable from the view point of quantum integrable systems. An important ingredient is the existence of the combinatorial Yang-Baxter equation. I then explain to what extent we can generalize this situation summarizing known results such as Kirillov-Reshetikhin crystals, path to rigged configuration bijection, and so on. If time permits, I also explain a recent work on new solutions of the combinatorial reflection equation, which enables us to formulate the box-ball system with boundary.

Hayate Suda* (Keio University)

Relationships between two linearizations of the box-ball system: rigged configuration and slot decomposition

It is known that the dynamics of the box-ball system can be linearized by several methods. In this talk, we give the explicit relationships between two different methods, the rigged configuration (RC) and the slot decomposition (SD). To do so, we introduce the notion of the carrier process with seat numbers and the corresponding seat number configuration (SC). We show that SC itself gives a new linearization method, and that by translating RC and SD in the language of SC, the relationships between RC and SD become clear. This talk is based on the joint work with Matteo Mucciconi, Tomohiro Sasamoto and Makiko Sasada.

Kazuki Maeda (University of Fukuchiyama)

Box-ball systems and biorthogonal polynomials

It is known that a time evolution equation of the box-ball systems is derived from the finite discrete Toda lattice and its generalizations. These equations and particular solutions are derived from the spectral transformations of biorthogonal polynomials and their reductions. In this talk, I will explain the theory of biorthogonal polynomials and its connection to the box-ball systems, which includes the known multicolor box-ball system and another multicolor box-ball system following different rules. This talk is based on:
K. Maeda, J. Phys. A: Math. Theor. 50 (2017) 365204,
K. Maeda, J. Integrable Syst. 3 (2018) xyy007.

Neil O'Connell* (University College Dublin)

Toda and probability

This talk will be a review of some probabilistic aspects of the open Toda chain, and how it relates to various generalisations of Pitman's $2M - X$ theorem and the Robinson-Schensted correspondence. I will also briefly describe some more recent developments in a non-Abelian setting.

Jeremy Quastel* (University of Toronto)

Polynuclear growth and the Toda lattice

The polynuclear growth model is one of the most important models in the KPZ universality class. Generally it has been studied in the droplet geometry, where it is equivalent to the longest increasing subsequence of a random permutation, whose solution sparked the KPZ revolution. We study it for general initial data and show that it is an integrable Markov process sharing the key structures of the KPZ fixed point, determinantal formulas for the transition probabilities and fixed time n-point distributions governed by completely integrable equations, the non-Abelian 2D Toda lattice. Joint with Konstantin Matetski and Daniel Remenik.

Tomohiro Sasamoto* (Tokyo Institute of Technology)

BBS like behaviors in skew RSK dynamics

In [1] we found a connection between q-Whittaker and skew Schur measures by using a generalization of RSK (Robinson-Schensted-Knuth) correspondence, which we called the skew RSK dynamics. An interesting observation was that this dynamics shows behaviors which look similar to BBS. In this talk we start from introducing the dynamics and explain the similarities.
[1] T. Imamura, M. Mucciconi, T. Sasamoto, *Skew RSK dynamics: Greene invariants, affine crystals and applications to q-Whittaker polynomials*, arXiv: 2106.11922

Ivan Corwin (Columbia University)

Invariant measures for KPZ systems with boundaries

Ralph Willox (University of Tokyo)

Direct and inverse scattering for the ultradiscrete KdV equation

In this talk I will describe a method for solving the initial value problem for the ultradiscrete KdV (udKdV) equation over the real numbers, which includes the initial value problem for the famous Takahashi-Satsuma Box & Ball system as a special case. The method is an ultradiscrete analogue of the inverse scattering transform for the usual, continuous, KdV equation.

In particular, I will first explain how to obtain the solution to a direct scattering problem for the udKdV equation for any potential, over the real numbers, with compact support, by explicitly constructing bound state and non-bound state eigenfunctions. I will then show how to reconstruct the potential in the scattering problem at different time steps which satisfy the udKdV evolution, using an ultradiscrete analogue of a Darboux dressing transformation. This reconstruction is based on data that uniquely characterise the soliton content and the so-called 'background' part in the initial potential, data which are obtained from the initial potential by successive Darboux undressing transformations.

This talk is based on the paper: "Darboux dressing and undressing for the ultradiscrete KdV equation" J.J.C. Nimmo, C.R. Gilson and R. Willox, J. Phys. A: Math. Theor. 52 (2019) 445201 (36pp).

Hanbaek Lyu (University of Wisconsin – Madison)

Scaling limit of soliton statistics of a multicolor box-ball system

The box-ball systems (BBS) are integrable cellular automata whose long-time behavior is characterized by the soliton solutions, and have rich connections to other integrable systems such as Korteweg-de Veris equation. Probabilistic analysis of BBS is an emerging topic in the field of integrable probability, which often reveals novel connection between the rich integrable structure of BBS and probabilistic phenomena such as phase transition and invariant measures. In this talk, we give an overview on the recent development in scaling limit theory of the classical Takahashi-Satsuma BBS as well as the multicolor BBS with one-sided random initial configurations. The integrability of BBS in this setting allows one to read-off the final soliton statistics directly from the initial configuration through various combinatorial operations. For the Takahashi-Satsuma case, these include hill-flattening operations of carrier process for soliton numbers and Pitman's water level process for soliton lengths. For the multicolor case, we use a modified Greene-Kleitman invariants for BBS, circular exclusion processes, Kerov–Kirillov–Reshetikhin bijection, combinatorial R, and Thermodynamic Bethe Ansatz to extract the corresponding soliton statistics.

Minmin Wang* (University of Sussex)

Soliton speeds in the box-ball system

In this talk, I'll focus on the system of linear equations for the effective soliton speeds in the box-ball system that has been obtained in [Ferrari, Nguyen, Rolla and Wang 2021]. I'll discuss the soliton dynamics that give rise to these equations as well as the mathematical tools that can turn the intuitive picture into rigorous arguments. Based on joint work with Pablo Ferrari, Chi Nguyen and Leo Rolla.

Chiara Franceschini (University of Modena)

Some results for hard rods of inhomogeneous size

A rod is an elongated particle which travels at a certain constant speed in absence of other rods. When two rods collide they exchange their position and continue their motion with the initial speed. The model was introduced by Dobrushin in the 50's for rods of identical length. Here we consider rods of different size and, in the spirit of the work of Boldrighini, Dobrushin and Sukhov of 1982, we show the hydrodynamic equation starting from the microscopic particle system. The key tool is a function which maps the hard rods system into the free gas of ballistic particles ignoring each other. It turns out that the Cauchy problem and the effective velocity have the same form of those for the soliton gas.

Benjamin Doyon (King's College London)

Generalised hydrodynamics for classical particle systems

I will overview the framework of generalised hydrodynamics (GHD), in the context of classical particle systems. I will emphasise the point of view based on the mapping to asymptotic coordinates via "real-space Bethe ansatz equations", which naturally leads to both the (generalised) thermodynamics and hydrodynamics of integrable systems and is particularly relevant in the box-ball, Toda and similar systems. I will explain how such ideas give a new, rather elementary derivation of the thermodynamics of the Toda system. If time permits, I will also overview how they lead to a derivation of the GHD equations for the quantum Lieb-Liniger model. I will finally emphasise how the GHD framework gives many exact predictions, such as large-scale correlation functions and diffusive corrections.

Herbert Spohn* (Technical University Munich)

Spacetime correlations of the classical Toda chain in thermal equilibrium

For integrable many-body systems one expects that, in leading order, equilibrium spacetime correlations scale ballistically with smooth shape functions. We report on novel results for the classical Toda lattice. (i) Correlations are obtained microscopically through molecular dynamics simulations over a wide range of equilibrium paramentes. (ii) These results are compared with predictions from the Landau-Lifshitz theory based on generalized hydrodynamics.

Vincent Pasquier (Université Paris Saclay)

Hydrodynamics of a box-ball system

I present the box-ball model, a cellular automaton which describes the motion of balls arranged on a line. It results from the discretization of soliton equations on the one hand and the crystal limit of the six vertex model on the other hand. This talk will mainly concentrate on the hydrodynamics of the model in connection with the so called generalized hydrodynamics. I will briefly review the thermodynamics of randomly placed balls and the corresponding stationary states so called GGE states. I will then show how linearized hydrodynamics works in these models and enables to make predictions in remarkable agreement with numerical simulations. Work in collaboration with Grégoire Misguich and Atsuo Kuniba.