Tridiagonal pairs and $U_q(\widehat{\mathfrak{sl}}_2)$

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Abstract

We will discuss a linear algebraic object called a **tridiagonal pair**.

Roughly speaking, this is a pair of diagonalizable linear maps on a finite-dimensional vector space, that each act on the eigenspaces of the other one in a (block) tridiagonal fashion.

We will describe the basic features of a tridiagonal pair, including the **shape vector**, the **tridiagonal relations**, and the **tetrahedron diagram**.

We will then use a tridiagonal pair to construct two modules for the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$.

This talk is based on joint work with Tatsuro Ito and Kenichiro Tanabe.

The definition of a tridiagonal pair

We now define a tridiagonal pair.

Let $\mathbb{F}$ denote a field.

Let V denote a vector space over $\mathbb{F}$ with finite positive dimension.

Let $\mathrm{End}(V)$ denote the $\mathbb{F}$-algebra consisting of the $\mathbb{F}$-linear maps from V to V.

Consider an ordered pair A, A^* of maps in $\mathrm{End}(V)$.

The definition of a tridiagonal pair

The above pair A, A^* is called a **tridiagonal pair** (or **TD pair**) whenever:

(i) each of A, A^* is diagonalizable;

(ii) there exists an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d), \qquad (1)$$

where $V_{-1} = 0$ and $V_{d+1} = 0$;

(iii) there exists an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^* such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta),$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$;

(iv) there does not exist a subspace $W \subseteq V$ such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

Comments

According to a common notational convention, A^* denotes the conjugate-transpose of A.

We are not using this convention.

In a TD pair, the elements A and A^* are arbitrary subject to (i)–(iv) above.

History and connections

The TD pairs were introduced in 1999 by Ito, Tanabe, and Terwilliger.

The TD pairs over an algebraically closed field were classified up to isomorphism by Ito, Nomura, and Terwilliger (2011).

TD pairs are related to:

- Q-polynomial distance-regular graphs,
- the orthogonal polynomials of the Askey scheme,
- the Askey-Wilson, Onsager, and q-Onsager algebras,
- the double affine Hecke algebra of type $(C_1^\vee, C_1)$,
- the Lie algebras $\mathfrak{sl}_2$ and $\widehat{\mathfrak{sl}}_2$,
- the quantum groups $U_q(\mathfrak{sl}_2)$ and $U_q(\widehat{\mathfrak{sl}}_2)$,
- integrable models in statistical mechanics.

Tridiagonal systems

In our study of tridiagonal pairs, it is useful to define a related object called a **tridiagonal system**.

Before defining this object, we recall some concepts.

Standard orderings

Let A, A^* denote a TD pair on V. An ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A is called **standard** whenever it satisfies (1).

If the ordering $\{V_i\}_{i=0}^d$ is standard then the inverted ordering $\{V_{d-i}\}_{i=0}^d$ is also standard, and no further ordering is standard.

Similar comments apply to A^*.

Given an eigenspace W of a diagonalizable linear map, the corresponding **primitive idempotent** acts on W as the identity map, and acts on the other eigenspaces as zero.

Definition

By a **tridiagonal system** (or **TD system**) on V, we mean a sequence

$$\Phi = (A, \{E_i\}_{i=0}^{d}, A^*, \{E_i^*\}_{i=0}^{\delta})$$

such that

(i) A, A^* is a TD pair on V;

(ii) $\{E_i\}_{i=0}^{d}$ is a standard ordering of the primitive idempotents of A;

(iii) $\{E_i^*\}_{i=0}^{\delta}$ is a standard ordering of the primitive idempotents of A^*.

The D_4 action

Consider a TD system $\Phi = (A, \{E_i\}_{i=0}^{d}, A^*, \{E^*_i\}_{i=0}^{\delta})$ on V.

Each of the following is a TD system on V:

$$\Phi^* = (A^*, \{E^*_i\}_{i=0}^{\delta}, A, \{E_i\}_{i=0}^{d});$$
$$\Phi^{\downarrow} = (A, \{E_i\}_{i=0}^{d}, A^*, \{E^*_{\delta-i}\}_{i=0}^{\delta});$$
$$\Phi^{\Downarrow} = (A, \{E_{d-i}\}_{i=0}^{d}, A^*, \{E^*_{\delta}\}_{i=0}^{\delta}).$$

The D_4 action, cont.

Viewing $*, \downarrow, \Downarrow$ as permutations on the set of all TD systems,

$$*^2 = 1, \qquad \downarrow^2 = 1, \qquad \Downarrow^2 = 1,$$
$$\Downarrow * = * \downarrow, \qquad \downarrow * = * \Downarrow, \qquad \downarrow\Downarrow = \downarrow\Downarrow .$$

The group generated by the symbols $*, \downarrow, \Downarrow$ subject to the above relations is called the **dihedral group** D_4. Recall that D_4 is the group of symmetries of a square, and has 8 elements.

The elements $*$, $\downarrow$, $\Downarrow$ induce an action of D_4 on the set of all TD systems.

TD systems in the same D_4-orbit are called **relatives**.

TD systems

Until further notice, fix a TD system on V:

$$\Phi = (A, \{E_i\}_{i=0}^{d}, A^*, \{E_i^*\}_{i=0}^{\delta}).$$

The triple product relations

Lemma

We have

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1; \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \qquad (0 \le i, j \le \delta),$$

$$E_i A^* E_j = \begin{cases} 0 & \text{if } |i-j| > 1; \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \qquad (0 \le i, j \le d).$$

The above equations are called the **triple product relations**.

The eigenvalues

Definition

For $0 \leq i \leq d$ let θ_i denote the eigenvalue of A corresponding to E_i. For $0 \leq i \leq \delta$ let θ^*_i denote the eigenvalue of A^* corresponding to E^*_i.

Definition

We call $\{\theta_i\}_{i=0}^{d}$ (resp. $\{\theta^*_i\}_{i=0}^{\delta}$) the **eigenvalue sequence** (resp. **dual eigenvalue sequence**) of Φ.

Showing $d = \delta$

Our next general goal is to show that $d = \delta$.

Definition

For $0 \leq i \leq \delta$ and $0 \leq j \leq d$ define

$$V_{i,j} = (E_0^* V + E_1^* V + \cdots + E_i^* V) \cap (E_0 V + E_1 V + \cdots + E_j V).$$

For example,

$$V_{\delta,0} = E_0 V, \qquad V_{0,d} = E_0^* V.$$

Showing $d = \delta$, cont.

Definition

Define

$$D = \min\{i + j | V_{i,j} \neq 0\}.$$

We have $D \leq \delta$, since $V_{\delta,0} \neq 0$.

We have $D \leq d$, since $V_{0,d} \neq 0$.

Abbreviate

$$U_i = V_{i,D-i} \qquad (0 \leq i \leq D).$$

Showing $d = \delta$, cont.

Lemma

(i) $(A^* - \theta^*_i I)U_i \subseteq U_{i-1}$ *for* $1 \leq i \leq D$;

(ii) $(A^* - \theta^*_0 I)U_0 = 0$;

(iii) $(A - \theta_{D-i} I)U_i \subseteq U_{i+1}$ *for* $0 \leq i \leq D - 1$;

(iv) $(A - \theta_0 I)U_D = 0$.

Showing $d = \delta$, cont.

Lemma

The sum $\sum_{i=0}^{D} U_i$ is:

- *nonzero;*
- *invariant under A and A^*;*
- *equal to V;*
- *contained in $E_0^* V + \cdots + E_D^* V$;*
- *contained in $E_0 V + \cdots + E_D V$.*

Moreover $D = d = \delta$.

The sum $V = \sum_{i=0}^{d} U_i$ is direct

Our next goal is to show that the sum $V = \sum_{i=0}^{d} U_i$ is direct.

Lemma

For $1 \le i \le d$,

- $(E^*_0 V + \cdots + E^*_{i-1} V) \cap (E_0 V + \cdots + E_{d-i} V) = 0$;
- $U_0 + \cdots + U_{i-1} \subseteq E^*_0 V + \cdots + E^*_{i-1} V$;
- $U_i \subseteq E_0 V + \cdots + E_{d-i} V$;
- $(U_0 + \cdots + U_{i-1}) \cap U_i = 0$.

Moeover the sum $V = \sum_{i=0}^{d} U_i$ *is direct.*

The Φ-split decomposition of V

The next result shows how the $\{U_i\}_{i=0}^d$ are related to the eigenspaces of A and A^*.

Lemma

For $0 \leq i \leq d$,

- $U_0 + \cdots + U_i = E_0^* V + \cdots + E_i^* V$;
- $U_i + \cdots + U_d = E_0 V + \cdots + E_{d-i} V$.

Definition

We call the sequence $\{U_i\}_{i=0}^d$ the **Φ-split decomposition** of V.

The Φ-split decomposition

Our next general goal is to show that for $0 \leq i \leq d$, the subspaces

$$\begin{array}{lll} E_i V, & E_i^* V, & U_i, \\ E_{d-i} V, & E_{d-i}^* V, & U_{d-i} \end{array}$$

all have the same dimension.

The projections $\{F_i\}_{i=0}^d$

For $0 \le i \le d$ define $F_i \in \mathrm{End}(V)$ such that

$$(F_i - I)U_i = 0,$$
$$F_i U_j = 0 \quad \text{if } i \neq j \quad (0 \le j \le d).$$

Thus F_i **is the projection from** V **onto** U_i. By linear algebra,

$$F_i F_j = \delta_{i,j} F_i \qquad (0 \le i, j \le d),$$
$$I = \sum_{i=0}^{d} F_i.$$

The projections $\{F_i\}_{i=0}^{d}$, cont.

Lemma

For $0 \leq i < j \leq d$,

$$E_j^* F_i = 0, \qquad F_j E_i^* = 0, \qquad E_{d-i} F_j = 0, \qquad F_i E_{d-j} = 0.$$

The projections $\{F_i\}_{i=0}^d$, cont.

Lemma

For $0 \leq i \leq d$,

$$F_i E_i^* F_i = F_i, \qquad E_i^* F_i E_i^* = E_i^*,$$
$$F_i E_{d-i} F_i = F_i, \qquad E_i F_{d-i} E_i = E_i.$$

The projections $\{F_i\}_{i=0}^d$, cont.

Lemma

The following hold for $0 \leq i \leq d$.

(i) *The linear maps*

$$\begin{array}{ccc} U_i & \to & E_i^* V \\ v & \mapsto & E_i^* v \end{array} \qquad \begin{array}{ccc} E_i^* V & \to & U_i \\ v & \mapsto & F_i v \end{array}$$

are bijections, and moreover, they are inverses.

(ii) *The linear maps*

$$\begin{array}{ccc} U_i & \to & E_{d-i} V \\ v & \mapsto & E_{d-i} v \end{array} \qquad \begin{array}{ccc} E_{d-i} V & \to & U_i \\ v & \mapsto & F_i v \end{array}$$

are bijections, and moreover, they are inverses.

The shape vector $\{\rho_i\}_{i=0}^d$

Definition

For $0 \leq i \leq d$ define $\rho_i = \dim U_i$.

Lemma

For $0 \leq i \leq d$ the following subspaces have dimension ρ_i:

$$E_i V, \qquad E_{d-i} V, \qquad E_i^* V, \qquad E_{d-i}^* V.$$

Moeover, $\rho_i = \rho_{d-i}$.

Definition

The sequence $\{\rho_i\}_{i=0}^d$ is called the **shape vector** of Φ.

The shape vector of Φ

We have shown that the shape vector $\{\rho_i\}_{i=0}^d$ is symmetric:

$$\rho_i = \rho_{d-i} \qquad (0 \leq i \leq d).$$

Our next goal is to show that $\{\rho_i\}_{i=0}^d$ is unimodal:

$$\rho_{i-1} \leq \rho_i \qquad (1 \leq i \leq d/2).$$

The maps L and R

Definition

Define

$$L = A^* - \sum_{i=0}^{d} \theta_i^* F_i,$$

$$R = A - \sum_{i=0}^{d} \theta_{d-i} F_i.$$

The maps L and R, cont.

Lemma

For $0 \le i \le d$ the following hold on U_i:

$$L = A^* - \theta_i^* I, \qquad R = A - \theta_{d-i} I.$$

The maps L and R, cont.

Lemma

(i) $LU_i \subseteq U_{i-1}$ *for* $1 \le i \le D$;

(ii) $LU_0 = 0$;

(iii) $RU_i \subseteq U_{i+1}$ *for* $0 \le i \le D-1$;

(iv) $RU_D = 0$.

Definition

We call L (resp. R) the **lowering map** (resp. **raising map**) of Φ.

The maps L and R, cont.

Lemma

The following hold for $0 \leq i \leq j \leq d$.

- *The map $R^{j-i} : U_i \to U_j$ is an injection if $i + j \leq d$, a bijection if $i + j = d$, and a surjection if $i + j \geq d$.*
- *The map $L^{j-i} : U_j \to U_i$ is an injection if $i + j \geq d$, a bijection if $i + j = d$, and a surjection if $i + j \leq d$.*

(Caution: the maps $R^{j-i} : U_i \to U_j$ and $L^{j-i} : U_j \to U_i$ are not inverses in general, even in the case $i + j = d$).

The shape vector $\{\rho_i\}_{i=0}^d$ is unimodal

Corollary

The shape vector $\{\rho_i\}_{i=0}^d$ satisfies $\rho_{i-1} \leq \rho_i$ for $1 \leq i \leq d/2$.

The tridiagonal relations

Our next goal is to show that the TD pair A, A^* satisfies a pair of cubic polynomial equations, called the **tridiagonal relations**.

First we assemble some preliminary results.

The tridiagonal relations; preliminaries

Let M denote the subalgebra of $\mathrm{End}(V)$ generated by A.

The vector space M has a basis $\{A^i\}_{i=0}^d$ and a basis $\{E_i\}_{i=0}^d$.

We now consider some elements in MA^*M.

The tridiagonal relations; preliminaries

Lemma

For $0 \leq i \leq d-1$,

$$E_i A^* E_{i+1} - E_{i+1} A^* E_i = (E_0 + \cdots + E_i) A^* - A^* (E_0 + \cdots + E_i).$$

Lemma

We have

$$\mathrm{Span}\{XA^*Y - YA^*X \mid X, Y \in M\} = \{ZA^* - A^*Z \mid Z \in M\}.$$

Theorem (Ito, Tanabe, Ter 2001)

There exists a sequence of scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ taken from $\mathbb{F}$ such that both

$$0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*],$$
$$0 = [A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^* A].$$

The sequence is unique if $d \geq 3$.

We are using the notation $[r, s] = rs - sr$.

The above relations are called the **tridiagonal relations**.

Proof Sketch: Assume that $d \geq 3$; otherwise the result is routine. By the previous lemma with $X = A^2$ and $Y = A$, there exists $Z \in M$ such that

$$A^2 A^* A - A A^* A^2 = Z A^* - A^* Z. \qquad (2)$$

There exists a polynomial $f(x)$ that has degree at most d such that $Z = f(A)$. Let each side of (2) act on the subspaces $\{U_i\}_{i=0}^d$. Comparing the actions of the two sides, we find that the degree of $f(x)$ is equal to 3. By these comments

$$A^2 A^* A - A A^* A^2 = \alpha_1 (A A^* - A^* A) \; + \; \alpha_2 (A^2 A^* - A^* A^2) \\ + \alpha_3 (A^3 A^* - A^* A^3)$$

for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}$ with $\alpha_3 \neq 0$. After a change of variables

$$\beta = \alpha_3^{-1} - 1, \qquad \gamma = -\alpha_2 \alpha_3^{-1}, \qquad \varrho = -\alpha_1 \alpha_3^{-1}$$

We get the first tridiagonal relation.

Applying an affine transformation to A and A^*

The tridiagonal relations can be adjusted by applying an affine transformation

$$A \mapsto \xi A + \zeta I, \qquad\qquad A^* \mapsto \xi^* A^* + \zeta^* I,$$

with $\xi, \zeta, \xi^*, \zeta^* \in \mathbb{F}$ and ξ, ξ^* nonzero.

Three special cases of the tridiagonal relations

Over the next few slides we display three special cases of the tridiagonal relations.

Special case: the Dolan-Grady relations

Example

For $\beta = 2$, $\gamma = \gamma^* = 0$, $\varrho = \varrho^* = 4$ the tridiagonal relations become the **Dolan/Grady relations**

$$[A, [A, [A, A^*]]] = 4[A, A^*],$$
$$[A^*, [A^*, [A^*, A]]] = 4[A^*, A]$$

where $[r, s] = rs - sr$.

The Dolan/Grady relations are the defining relations for the **Onsager Lie Algebra**.

Special case: the q-Serre relations

Example

For $\beta = q^2 + q^{-2}$, $\gamma = \gamma^* = 0$, $\varrho = \varrho^* = 0$ the tridiagonal relations become the **q-Serre relations**

$$[A, [A, [A, A^*]_q]_{q^{-1}}] = 0,$$
$$[A^*, [A^*, [A^*, A]_q]_{q^{-1}}] = 0$$

where

$$[r, s] = rs - sr, \qquad [r, s]_q = qrs - q^{-1}sr.$$

The above q-Serre relations are the defining relations for the **positive part** U_q^+ of the **quantum group** $U_q(\widehat{\mathfrak{sl}}_2)$.

Example

For $\beta = q^2 + q^{-2}$, $\gamma = \gamma^* = 0$, $\varrho = \varrho^* = -(q^2 - q^{-2})^2$ the tridiagonal relations become the **q-Dolan/Grady relations**

$$[A, [A, [A, A^*]_q]_{q^{-1}}] = (q^2 - q^{-2})^2 [A^*, A],$$
$$[A^*, [A^*, [A^*, A]_q]_{q^{-1}}] = (q^2 - q^{-2})^2 [A, A^*].$$

The q-Dolan/Grady relations are the defining relations for the **q-Onsager algebra** O_q.

The eigenvalue and dual eigenvalue sequence

We return our attention to the tridiagonal relations in the general case.

An analysis of these relations reveals that the eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta^*_i\}_{i=0}^d$ each satisfy a three-term recurrence.

The details are on the next slide.

The eigenvalue and dual eigenvalue sequence

Theorem (Ito, Tanabe, Ter 2001)

Given scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ that satisfy the tridiagonal relations. Then each of

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}$$

is equal to $\beta + 1$ for $2 \leq i \leq d-1$. Moreover,

$$\begin{aligned}
\gamma &= \theta_{i-1} - \beta\theta_i + \theta_{i+1} \qquad (1 \leq i \leq d-1),\\
\gamma^* &= \theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1} \qquad (1 \leq i \leq d-1),\\
\varrho &= \theta^2_{i-1} - \beta\theta_{i-1}\theta_i + \theta^2_i - \gamma(\theta_{i-1} + \theta_i) \qquad (1 \leq i \leq d),\\
\varrho^* &= \theta^{*2}_{i-1} - \beta\theta^*_{i-1}\theta^*_i + \theta^{*2}_i - \gamma^*(\theta^*_{i-1} + \theta^*_i) \qquad (1 \leq i \leq d).
\end{aligned}$$

The tetrahedron diagram

Recall our TD system on V:

$$\Phi = (A, \{E_i\}_{i=0}^{d}, A^*, \{E_i^*\}_{i=0}^{d}).$$

Our next general goal is to describe the **tetrahedron diagram** for Φ.

Decompositions

We sharpen our notation.

Definition

By a **decomposition of V of length d**, we mean a sequence $\{W_i\}_{i=0}^{d}$ of nonzero subspaces whose direct sum is V.

Examples of decompositions

Example

Each of the following form a decomposition of V:

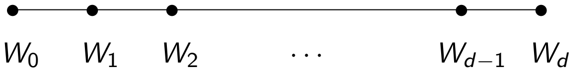
(i) the eigenspaces $\{E_i V\}_{i=0}^d$ of A;

(ii) the eigenspaces $\{E_i^* V\}_{i=0}^d$ of A^*;

(iii) the subspaces $\{U_i\}_{i=0}^d$.

Decompositions and diagrams

Let $\{W_i\}_{i=0}^d$ denote a decomposition of V. We describe this decomposition by the diagram



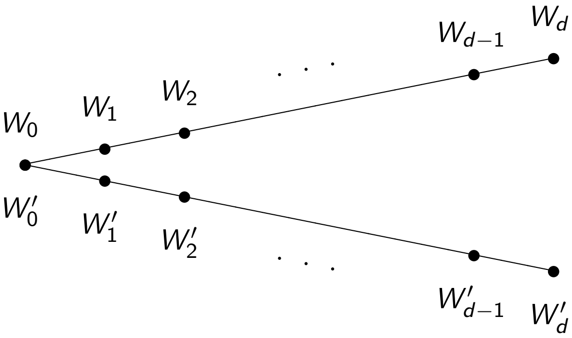
The labels W_i might be suppressed, if they are clear from the context.

More diagrams

Let $\{W_i\}_{i=0}^d$ and $\{W'_i\}_{i=0}^d$ denote decompositions of V. The condition

$$W_0 + W_1 + \cdots + W_i = W'_0 + W'_1 + \cdots + W'_i \qquad (0 \le i \le d)$$

will be described by the diagram



The Φ-split decomposition, revisited

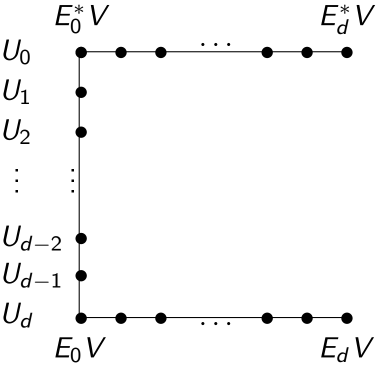
Recall the Φ-split decomposition $\{U_i\}_{i=0}^d$ of V.

Earlier we showed that for $0 \le i \le d$,

$$U_0 + \cdots + U_i = E_0^* V + \cdots + E_i^* V,$$
$$U_i + \cdots + U_d = E_0 V + \cdots + E_{d-i} V.$$

The corresponding diagram is shown on the next slide.

The Φ-split decomposition, revisited

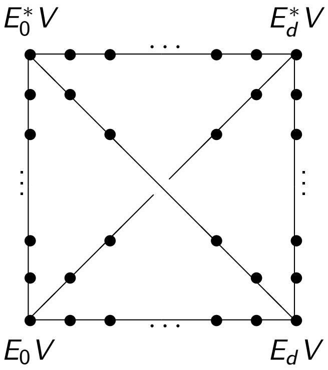


The relatives of Φ

Referring to the Φ-split decomposition of V, we can replace Φ by its relatives to get additional decompositions of V.

These decompositions are shown on the next slide.

The tetrahedron diagram



The above diagram is called the **tetrahedron diagram** of Φ.

Eigenspace decompositions

In the tetrahedron diagram, the horizontal decompositions are the eigenspace decompositions for A and A^*.

It is tempting to view the other four decompositions as eigenspace decompositions for some other maps in $\mathrm{End}(V)$.

To make progress in this direction, we impose an assumption on Φ.

For the rest of this talk, fix a nonzero $q \in \mathbb{F}$ that is not a root of unity.

Recall the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad n = 0, 1, 2, \ldots$$

Definition

The TD system Φ is said to have **q-geometric type** whenever

$$\theta_i = q^{2i-d}, \qquad \theta^*_i = q^{d-2i}$$

for $0 \le i \le d$.

TD systems of q-geometric type

For the rest of this talk, assume that Φ has q-geometric type.

In this case

$$\beta = q^2 + q^{-2}, \qquad \gamma = \gamma^* = 0, \qquad \varrho = \varrho^* = 0.$$

Moreover the TD relations become the q-Serre relations

$$[A, [A, [A, A^*]_q]_{q^{-1}}] = 0,$$
$$[A^*, [A^*, [A^*, A]_q]_{q^{-1}}] = 0.$$

A decomposition and the corresponding map

We bring in some notation.

Definition

Let $\{W_i\}_{i=0}^{d}$ denote a decomposition of V.

The **corresponding map** is the element $M \in \mathrm{End}(V)$ such that $(M - q^{d-2i}I)W_i = 0$ for $0 \leq i \leq d$.

Thus for $0 \leq i \leq d$, the subspace W_i is an eigenspace of M with eigenvalue q^{d-2i}.

A decomposition and the corresponding map

Lemma

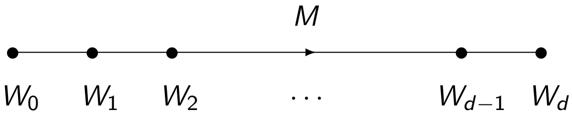
For a decomposition $\{W_i\}_{i=0}^d$ of V the following maps are inverses:

(i) *the map corresponding to $\{W_i\}_{i=0}^d$;*

(ii) *the map corresponding to $\{W_{d-i}\}_{i=0}^d$.*

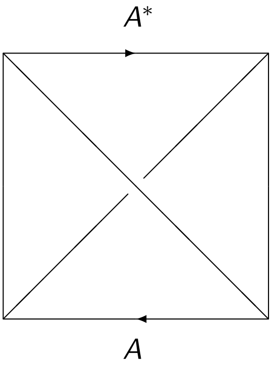
A decomposition and the corresponding map

For a decomposition $\{W_i\}_{i=0}^d$ of V, the corresponding map M is described by the diagram



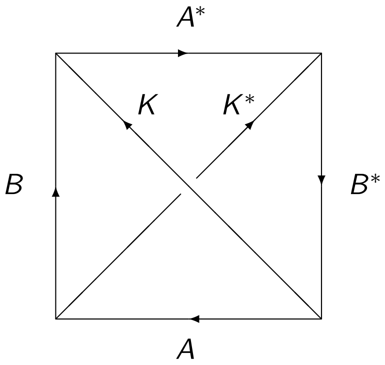
We might suppress the labels W_i along with the • notation, if they are clear from the context.

For example:



The maps B, B^*, K, K^*

We define the linear maps B, B^*, K, K^* as follows:



Some relations involving A, A^*, B, B^*, K, K^*

Over the next few slides, we give some relations involving A, A^*, B, B^*, K, K^*.

Lemma

We have

$$\frac{qAB - q^{-1}BA}{q - q^{-1}} = I,$$

$$\frac{qBA^* - q^{-1}A^*B}{q - q^{-1}} = I,$$

$$\frac{qA^*B^* - q^{-1}B^*A^*}{q - q^{-1}} = I,$$

$$\frac{qB^*A - q^{-1}AB^*}{q - q^{-1}} = I.$$

Lemma

We have

$$\frac{qK^{-1}A - q^{-1}AK^{-1}}{q - q^{-1}} = I,$$

$$\frac{qBK^{-1} - q^{-1}K^{-1}B}{q - q^{-1}} = I,$$

$$\frac{qKA^* - q^{-1}A^*K}{q - q^{-1}} = I,$$

$$\frac{qB^*K - q^{-1}KB^*}{q - q^{-1}} = I.$$

Some relations involving A, A^*, B, B^*, K, K^*, cont.

Lemma

We have

$$\frac{qAK^* - q^{-1}K^*A}{q - q^{-1}} = I,$$

$$\frac{qK^{*-1}B - q^{-1}BK^{*-1}}{q - q^{-1}} = I,$$

$$\frac{qA^*K^{*-1} - q^{-1}K^{*-1}A^*}{q - q^{-1}} = I,$$

$$\frac{qK^*B^* - q^{-1}B^*K^*}{q - q^{-1}} = I.$$

Lemma

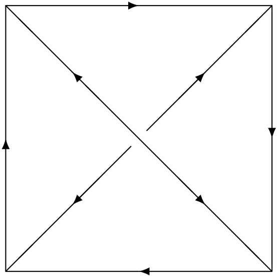
The maps B, B^* satisfy the q-Serre relations

$$[B,[B,[B,B^*]_q]_{q^{-1}}] = 0,$$
$$[B^*,[B^*,[B^*,B]_q]_{q^{-1}}] = 0.$$

We just gave many relations involving A, A^*, B, B^*, K, K^*.

These relations are summarized by the diagram below.

The relations in summary



- For any two arcs with the same endpoints and pointing in the opposite direction, the corresponding maps are inverses.
- For any two arcs that create a directed path of length two, the corresponding maps r, s satisfy $\frac{qrs - q^{-1}sr}{q - q^{-1}} = 1$.
- For any two arcs that are distinct and parallel (horizontal or vertical), the corresponding maps satisfy the q-Serre relations.

Definition

Define the algebra $U_q(\widehat{\mathfrak{sl}}_2)$ by generators $e_i^{\pm}$, $K_i^{\pm 1}$, $i \in \{0, 1\}$ and the following relations:

$$
\begin{aligned}
K_i K_i^{-1} &= K_i^{-1} K_i = 1, \\
K_0 K_1 &= K_1 K_0, \\
K_i e_i^{\pm} K_i^{-1} &= q^{\pm 2} e_i^{\pm}, \\
K_i e_j^{\pm} K_i^{-1} &= q^{\mp 2} e_j^{\pm}, \qquad i \neq j, \\
[e_i^{+}, e_i^{-}] &= \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\
[e_0^{\pm}, e_1^{\mp}] &= 0,
\end{aligned}
$$

$$(e_i^{\pm})^3 e_j^{\pm} - [3]_q (e_i^{\pm})^2 e_j^{\pm} e_i^{\pm} + [3]_q e_i^{\pm} e_j^{\pm} (e_i^{\pm})^2 - e_j^{\pm} (e_i^{\pm})^3 = 0, \quad i \neq j.$$

We call $e_i^{\pm}$, $K_i^{\pm 1}$ the **Chevalley generators** for $U_q(\widehat{\mathfrak{sl}}_2)$

Lemma (Ito+Ter 2003)

The algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is isomorphic to the algebra with generators $y_i^{\pm}$, $k_i^{\pm 1}$, $i \in \{0,1\}$ and the following relations:

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \qquad k_0 k_1 \text{ is central},$$

$$\frac{q y_i^+ k_i - q^{-1} k_i y_i^+}{q - q^{-1}} = 1,$$

$$\frac{q k_i y_i^- - q^{-1} y_i^- k_i}{q - q^{-1}} = 1, \qquad \frac{q y_i^- y_i^+ - q^{-1} y_i^+ y_i^-}{q - q^{-1}} = 1,$$

$$\frac{q y_i^+ y_j^- - q^{-1} y_j^- y_i^+}{q - q^{-1}} = k_0^{-1} k_1^{-1}, \qquad i \neq j,$$

$$(y_i^{\pm})^3 y_j^{\pm} - [3]_q (y_i^{\pm})^2 y_j^{\pm} y_i^{\pm} + [3]_q y_i^{\pm} y_j^{\pm} (y_i^{\pm})^2 - y_j^{\pm} (y_i^{\pm})^3 = 0, \quad i \neq j.$$

We call $y_i^{\pm}$, $k_i^{\pm 1}$ the **equitable generators** *of $U_q(\widehat{\mathfrak{sl}}_2)$.*

Theorem (Ito+Ter, 2003)

We refer to our TD system Φ on V of q-geometric type.

The vector space V is an irreducible $U_q(\widehat{\mathfrak{sl}}_2)$-module on which the equitable generators act as follows:

generator	y_0^+	y_1^+	y_0^-	y_1^-	k_0	k_1	k_0^{-1}	k_1^{-1}
action on V	B^*	B	A^*	A	K	K^{-1}	K^{-1}	K

Moreoever, V is an irreducible $U_q(\widehat{\mathfrak{sl}}_2)$-module on which the equitable generators act as follows:

generator	y_0^+	y_1^+	y_0^-	y_1^-	k_0	k_1	k_0^{-1}	k_1^{-1}
action on V	A	A^*	B^*	B	K^*	K^{*-1}	K^{*-1}	K^*

THE END

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