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Symmetric tensor categories

Lectures 1 and 2: Algebra and representation theory without vector spaces.

Abstract: A modern view of representation theory is that it is a study not just of individual representations (say, finite dimensional representations of an affine group or, more generally, supergroup scheme G over an algebraically closed field k) but also of the category Rep(G) formed by them. The properties of Rep(G) can be summarized by saying that it is a symmetric tensor category (shortly, STC) which uniquely determines G. A STC is a natural home for studying any kind of linear algebraic structures (commutative algebras, Lie algebras, Hopf algebras, modules over them, etc.); for instance, doing so in Rep(G) amounts to studying such structures with a G-symmetry. It is therefore natural to ask: does the study of STC reduce to group representation theory, or is it more general? In other words, do there exist STC other than Rep(G)? If so, this would be interesting, since algebra in such STC would be a new kind of algebra, one "without vector spaces". Luckily, the answer turns out to be ``yes". I will discuss examples in characteristic zero and p>0, and also Deligne's theorem, which puts restrictions on the kind of examples one can have.

Lecture 3: Representation theory in non-integral rank.

Examples of symmetric tensor categories over complex numbers which are not representation categories of supergroups were given by Deligne-Milne in 1981. These very interesting categories are interpolations of representation categories of classical groups GL(n), O(n), Sp(n) to arbitrary complex values of n. Deligne later generalized them to symmetric groups and also to characteristic p, where, somewhat unexpectedly, one needs to interpolate n to p-adic integer values rather than elements of the ground field. I will review some of the recent results on these categories and discuss algebra and representation theory in them.

Lecture 4: Symmetric tensor categories of moderate growth and modular representation theory.

Deligne categories discussed in Lecture 3 violate an obvious necessary condition for a symmetric tensor category (STC) to have any realization by finite dimensional vector spaces (and in particular to be of the form Rep(G)): for each object X the length of the n-th tensor power of X grows at most exponentially with n. We call this property ``moderate growth". So, it is natural to ask if there exist STC of moderate growth other than Rep(G). In characteristic zero, the negative answer is given by the remarkable theorem of Deligne (2002), discussed in Lectures 1,2. Namely Deligne's theorem says that a STC of moderate growth can always be realized in supervector spaces. However, in characteristic p the situation is much more interesting. Namely, Deligne's theorem is known to fail in any characteristic p>0. The simplest exotic symmetric tensor category of moderate growth (i.e., not of the form Rep(G)) for p>3 is the semisimplification of the category of representations of Z/p, called the Verlinde category. For example, for p=5, this category has an object X such that X^2=X+1, so X cannot be realized by a vector space (as its dimension would have to equal the golden ratio). I will discuss some aspects of algebra in these categories, in particular failure of the PBW theorem for Lie algebras (and how to fix it) and a generalization of Deligne's theorem in characteristic p due to Kevin Coulembier, Victor Ostrik and myself. This generalization allows one to prove new properties of modular representations of finite groups (and, more generally, affine group schemes) which were previously out of reach. I will also discuss a family of non-semisimple exotic categories in characteristic p constructed in my joint work with Dave Benson and Victor Ostrik, and their relation to the representation theory of groups (Z/p)^n over a field of characteristic p.