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The first day of the summer school deals with basic concepts of nonlinear dynamics, with applications to synchronization of biological oscillations. A good background reference is Kaplan D, Glass L. *Understanding nonlinear dynamics.* Springer-Verlag, New York (1997). A review article that is light on technical details is [4]. Chapters 2, 3 and 5 in [3] are based on the material given in the lectures and contain many of the mathematical details.

The codes for the computer exercises are available at: `http://cube.cnd.mcgill.ca/ebook/index.html`. This computer code could form the starting point for the projects. You would need to significantly improve this code to make progress on the projects.

1 Iteration of one-dimensional finite-difference equations

This gives instructions for running the programs to study the quadratic map

$$x_{i+1} = \mu(1 - x_i)x_i \tag{1}$$

using Matlab.

There are 4 programs.

- **fditer('quadmap',xzero,mu,niter)** This program iterates the quadratic map. There are three input arguments: **xzero** is the initial condition; **mu** is the bifurcation parameter in Eq. 1; **niter** is the number of iterations. The output is a vector **y** of length **niter** containing the iterated values.

- **testper(y,epsilon,maxper)** This program determines if there is a periodic orbit in the sequence given by the vector **y** whose period is less than or equal to **maxper**. The convergence criterion is that two iterates of **y** are closer than **epsilon**. The output is the period **per**. If no convergence is found the output is **-1**.

- **bifurc('quadmap',mubegin,muend)** This plots the bifurcation diagram of the quadratic function for 100 steps of μ between **mubegin** and **muend**.

- **cobweb('quadmap',xzero,mu,nstep)** This program iterates the quadratic map. There are three input arguments: **xzero** is the initial condition; **mu** is the parameter; **nstep** is the number of iterations for which you will compute the cobweb.

1.1 How to run the programs

The following steps give an illustration example of how to run these programs.

1. Open up a Matlab window. All instructions are carried out in this window.

2. To generate 100 iterations of the quadratic map with an initial condition of $x_1 = 0.5$, $\mu = 3.973$ type
   `y=fditer('quadmap',0.5,3.973,100);`

3. To plot the time series from this iteration type
   `plot(y,'+');`

4. To determine if there is a period of length less than or equal to 20 with a convergence of 0.00001 of the time series **y** type
   `per=testper(y,.00001,20);` In this case there is no period and the program returns **per**$= -1$. If a value $\mu = 3.2$ had been used to generate the time series in the quadratic program, the program testper returns a value of **per**$= 2$.

5. To plot a cobweb diagram for the quadratic map with an initial condition of **xzero=0.3** and $\mu = \mathbf{3.825}$ with 12 steps, type
   `cobweb('quadmap',0.3,3.825,12);`

1.2 Exercises

The above programs essentially carry out Computer Projects 1-3 from Kaplan and Glass, pages 51-53 for the quadratic map. You are in a position to carry out the remainder of the Projects 4-5 using the quadratic map.

Project 4 asks you to compute the sequence of periodic orbits encountered as μ is increased. Try to find all ranges of μ that give periodic orbits up to period 6. As μ is increased you should be able to find windows that give periodic windows in the sequence 1,2,4,6,5,3,6,5,6,4,6,5,6. You will want to try to increment μ sufficiently finely to find the different periodic orbits. The sequence of periodic orbits is called the **universal** sequence. It is the same for all maps with a quadratic maximum and a single hump.

Project 5 asks you to determine Feigenbaum's number. Feigenbaum's number is defined as follows. Call Δ_n the range of μ values that give a period n orbit. Then Feigenbaum found that in a sequence of period-doubling bifurcations

$$\lim_{n \to \infty} \frac{\Delta_n}{\Delta_{2n}} = 4.4492\ldots$$

The constant, 4.6692 ... is now called **Feigenbaum's number**.

According to Feigenbaum, he initially discovered this number by carrying out numerical iterations on a hand calculator. As the period of the cycle gets longer, the range of parameter values over which a given period is found gets smaller. Therefore, it is necessary to think carefully about what is involved in the calculation. Try to numerically compute

$$\frac{\Delta_8}{\Delta_{16}}$$

and

$$\frac{\Delta_6}{\Delta_{12}}.$$

You will want to vary μ over a range of values. How should you locate the value of μ where the period changes?

The behaviors found for the quadratic map here are also found in other simple maps, complicated equations, and a variety of experimental systems. It is this **universal** behavior that has attracted the attention of physicists and others.

By making appropriate modifications in the Matlab programs, you can adapt the programs so that they carry out similar computations for the single-humped sine map

$$x_{i+1} = \mu \sin(\pi x_i), \tag{2}$$

where $0 < \mu < 1$.

2 Geometry of Fixed Points in Two-Dimensional Maps

This laboratory enables you to generate correlated random dot patterns. The laboratory is based on observations made in following papers.

Glass, L. Moiré effect from random dots. Nature 223, 578-580 (l969); Glass, L., R. Perez. Perception of random dot interference patterns. Nature 246, 360-362 (l973).

The programs show a random pattern of 400 dots superimposed on itself following a rescaling **a** in the x coordinate, **b** in the y coordinate, and a rotation by the angle θ. There is a fixed point at $x = y = 0$.

This transformation is given by the equations

$$x' = ax\cos\theta - by\sin\theta \tag{3}$$

$$y' = ax\sin\theta + by\cos\theta \tag{4}$$

The eigenvalues of this transformation are given by

$$\lambda_{\pm} = \frac{(a+b)\cos\theta \pm \sqrt{(a-b)^2 - ((a+b)\sin\theta})^2}{2} \tag{5}$$

The eigenvalues of this transformation can be related to the geometry of the transformation in the neighborhood of the fixed point at $x = y = 0$. If the eigenvalues are complex numbers, the fixed point is a focus, if the eigenvalues are real and are both inside or outside the unit circle, the fixed point is a node, if the eigenvalues are real and one is inside the unit circle, and the other is outside the unit circle, the fixed point is a saddle. If the eigenvalues are pure complex the fixed point is a center.

There is one program.

- **dots(a,b,thetam,numb)**. This program generates 400 random dots and **numb** iterates of each of these dots using the transformation above. (Use 4 iterates for better visualization but a single iterate would also suffice if it were possible to make big dots.) The dots are plotted, and the eigenvalues of the transformation are given underneath the figure.

2.1 How to run the program

The following steps give an illustration example of how to run these programs.

1. Open up a Matlab window. All instructions are carried out in this window.

2. To display a plot with $a = 0.95$, $b = 1.05$, and $\theta = 0.4/\pi$ type
   `dots(0.95,1.05,0.4/pi,4);`

2.2 Exercises

You may wish to see what happens for particular values of the parameters. Try to find parameters that give centers, focuses, nodes, and saddles. Try increasing the angle of rotation until you can no longer perceive the geometry of the transformation. Can you predict theoretically the critical parameters that destroy your ability to perceive the geometry? If so, this might be a good result in the field of visual perception.

Here is a problem. In general, it should be impossible to find a bifurcation from a saddle to a focus except in exceptional cases. Consider the bifurcations observed with $a = 0.95$, $b = 1.05$ as θ varies. Is there a direct bifurcation from a saddle to a focus? Try to determine this by looking at the pictures and analytically. Which is simpler and which is more informative?

3 Laboratory - Resetting curves for Poincaré oscillator

One of the simplest models of a limit cycle oscillation is the Poincaré oscillator. The equations for this model are

$$\begin{aligned} \frac{dr}{dt} &= kr(1-r), \\ \frac{d\phi}{dt} &= 2\pi, \end{aligned} \tag{6}$$

where k is a positive parameter. Starting at any value of r, except $r = 0$, there is an evolution until $r = 1$. The parameter k controls the relaxation rate. In this laboratory we consider the relaxation in the limit $k \to \infty$.

There are two programs in this laboratory.

- **resetmap(b)** This program computes the resetting curve (new phase versus old phase) for a stimulus strength **b**. The output is a matrix with 2 columns and 102 lines. There are two points just less than and just greater than $\phi = 0.5$. These points are needed especially for the case where $b > 1$.

- **poincare(phizer,b,tau,niter)** This program does an iteration of the periodically stimulated Poincaré oscillator, where `phizer` is the initial phase, **b** is the stimulation strength, **tau** is the period of the stimulation, and **niter** is the number of iterations. It is valid for $0 < \tau < 1$. The output consists of two arrays. `phi` is is a listing of the successive phases during the periodic stimulation. `beats` is a listing of the number of beats that occur between successive stimuli.

3.1 How to run the programs

1. Open a Matlab window. All the instructions are carried out in this window.

2. To compute the resetting curve for $b = 1.10$, you type
   ```
   [phi,phiprime]=resetmap(1.10);
   ```

3. To plot out the resetting curve just computed type
   ```
   plot(phi,phiprime,'*')
   ```

4. To simulate periodic stimulation of the Poincaré oscillator type
   ```
   [phi,beats]=poincare(.3,1.13,0.35,100);
   ```
   This will generate two time series of 100 iterates from an initial condition of $\phi = 0.3$, with $b = 1.13$, and $\tau = 0.35$. The array `phi` is the successive phases during the stimulation. The array `beats` is the number of beats between stimuli.

5. To display the output as a return map, type
   ```
   plot(phi(2:99),phi(3:100),'*')
   ```
   This plots out the successive phases of each stimulus as a function of the phase of the preceding stimulus. The points lie on a one-dimensional curve. The dynamics in this case are chaotic. In fact, what is observed here is very similar to what is actually observed during periodic stimulation of heart cell aggregates described in the first lecture of the course.

6. To display the number of beats between stimuli, type
   `plot(beats,'*')`

7. The rotation number gives the ratio between number of beats and the number of stimuli during a stimulation. This is the average number of beats per stimulius. To compute the rotation number type
   `sum(beats)/length(beats)`

3.2 Exercises

Try the following exercises.

1. Use the program **resetting** to compute the resetting curves for several values of **b** in the range from 0 to 2. In particular determine the value of **b** at which the topology of the resetting curve changes.

2. Determine whether or not the successive iterates of `phi` are periodic assuming different value of $(\mathbf{b},\tau)$ (use program `testper`). (i) What do you find for different values of b and τ? What is the ratio of the number of stimuli to the number of action potentials? (ii) Find values for which there are different asymptotic behaviors depending on the initial condition? (iii) Find values that give quasiperiodic dynamics (for nonzero **b**) (iv) Can you find a period-doubling route to chaos?

4 Research Level Projects

For students with an interest in phase locking, I suggest research level problems; strong results would merit publication. Students require advanced knowledge of nonlinear dynamics and strong computer skills to make progress on these problems. Please feel free to discuss these problems with me.

Interaction of biological oscillations with periodic inputs represent a fundamental problem that recurs in many contexts [4]. Our group has studied the effects of periodic stimuli on the heart [6, 7, 5, 8]. Several years ago, I worked with N. Q. Balaban at the Hebrew University who has studied the correlations of the cell cycle time between mother and daughter cells [1, 10].

I suggest two projects, but many aspects of the first project have been published in collaboration with Wilson Façanha and B. Oldeman [9]. The second relates to the work of Balaban and collaborators. If you make sufficient progress on the numerics, I could suggest directions that we still do not understand very well, where good new results (especially if they have new analytic insights) would be worthy of publication.

Project A. Entraining the 2D Poincaré Oscillator

Perhaps the simplest ordinary differential equation that has been used to model biological oscillations is the Poincaré oscillator (AKA the Radial Isochron Clock) [7, 5, 8]. In a polar (r, ϕ) coordinate system, the equations for this system are:

$$\frac{dr}{dt} = kr(1-r), \qquad \frac{d\phi}{dt} = 2\pi \tag{7}$$

The model for resetting the oscillator is to apply a δ function stimulus to the state point leading to an instantaneous increase of the x-coordinate by a magnitude b. Following the stimulus, the equations of motion take over and the analytic formula for the trajectory therefore be computed. Consequently, if a stimulus is given when the system is at state point (r', ϕ') we have

$$\begin{aligned} r'_i &= (r_i^2 + b^2 + 2br_i \cos 2\pi\phi_i)^{1/2}, \\ \phi'_i &= \frac{1}{2\pi} \arccos \frac{r_i \cos 2\pi\phi_i + b}{r'_i}. \end{aligned} \tag{8}$$

where (r'_i, ϕ'_i) are the coordinates immediately after the stimulus.

Consider periodic stimulation with a time interval of τ between stimuli. There are many different dynamical behaviors possible. One is called phase locking in which there are N cycle of the stimulus for each M cycles of the oscillator. But there can also be chaotic dynamics or quasiperiodic dynamics. The different dynamics are found for different ranges of the frequency and amplitude of the periodic δ function pulse.

The effects of periodic stimulation can be represented as a 2D map.

$$\begin{aligned} r_{i+1} &= \frac{r'_i}{(1-r'_i)\exp(-k\tau) + r'_i}, \\ \phi_{i+1} &= \phi'_i + \tau (\text{mod } 1). \end{aligned} \tag{9}$$

Your project is to compute the different locking regions as a function of (k, b, τ). What is the maximum number of fixed points and periodic points possible for any choice of parameters? For what parameters and initial conditions is there chaos? Can you prove chaos?

Considering the locking zones in (b, τ) for k fixed. What is the organization of the zones for some fixed value of k and how does picture evolve as k increases. The results for k infinite are in [7, 5, 8]. This is a fascinating research level problem that has been rarely studied. An early paper [5] left many questions open. A more recent paper answered some of them [9] but there is still a lot that could be done.

Algebraically, compute the boundaries (in the stimulus amplitude - stimulus period plane) along which the 1:1 phase locking rhythm of the Poincaré oscillator in the infinite relaxation limit ($k \to \infty$) loses stability and determine the type of bifurcation along the boundary.

Computer project. Determine the phase locking regions in the infinite relaxation limit ($k \to \infty$). Now do the same for the k finite.

Project B. Phase locking in fattened Arnold map

Balaban and colleagues [1, 10], proposed a model for the cell cycle that takes into account the cycle time of the mother cell and also the phase of another (for example the circadian) oscillator. It turns out that this model is the same with a change of variables to a model studied earlier called the fattened Arnold map [2].

The model for analysis is:

$$\begin{aligned} T_{n+1} &= T_0(1-\alpha) + \alpha T_n + k \sin\left(\frac{2\pi t_{n+1}}{T_{osc}}\right) && (10) \\ t_{n+1} &= T_n + t_n && (11) \end{aligned}$$

where T_n is the nth cycle time, t_n is the actual time of the nth division, α is a parameter $[-1 \leq \alpha \leq 1]$ that gives the mother's influence on the daughter's cycle time, $0 < k < 1$ scales the strength of the periodic input. T_{osc} is the period of the periodic input. When $\alpha = 0$ this is the degree 1 sine circle map as discussed for example in [4]. When $\alpha = 1$, the map is called the "standard map" which has been studied so much by physicists that it rates Wikipedia and Scholarpedia articles. Your job is to analyze the bifurcations in this map as a function of T_0, k and α. You might start with $\alpha = \pm 0.5$ and assume $T_{osc} = 24$ hours. The long-range goal would be to derive an analysis of the dynamics that takes into account the continuous changes in the locking zones and bifurcations in the space of 3 parameters similar to the type of analysis done for the problem in the first project [9].

References

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