Numerical Inclusion of Exact Periodic Solutions for Time Delay Duffing Equation

Shin'ichi OISHI
Faculty of Science and Engineering, Waseda University, JST CREST
3–4–5 Okubo, Shinjuku-ku, Tokyo, 269–8555 Japan

Constructive implicit function theorem is proposed for including one dimensional solution manifolds consisting of exact periodic solutions. Bounds for amplitudes and angle frequencies of periodic solutions are proved. Furthermore, conjecture for a lower bound for a number of periodic solutions is given when a time delay is given.

Autonomous Time Delay Duffing Equation

This talk consist of two parts. In the first place, the following autonomous time delay Duffing equation is treated:

$$\frac{d^2x(t)}{dt^2} + k\frac{dx(t)}{dt} + \Omega_0^2 x(t) + \gamma x(t)^3 + \alpha(x(t) - x(t-\tau)) = 0 \tag{1}$$

where $\tau > 0$ is delay. This equation is considered in Ref.[2]. Eq.(1) is obtained as a mathematical model of a mechanical system cutting a surface of rotating rounding rods. $x(t-\tau)$ is a shape of surface of one cycle before and $x(t)$ is a surface of the present time. In the following, we restrict ourselves to the parameter values considered in Ref.[2] as $k = 0.1, \Omega_0^2 = 2, \gamma = 0.25, \alpha = 0.25$. We will study the change of a set of periodic solutions as changing τ. Autonomous periodic oscillations arising in Eq.(1) are called chattering. Existence of chattering is known from hundreds years ago. Chattering is harmful when one wants to shaving the surface of rod smoothly. However, it is used to make periodic elegant texture in the traditional wood working and pottery named as a flying canna technique.

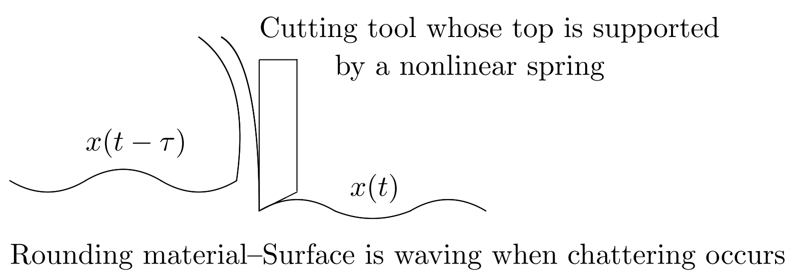

Figure 1: Mechanical cutting system described by the time delay Duffing equation. In this figure rounding surface is described by a plane by looking locally. The function $x(t-\tau)$ describes the surface cutted in one cycle before of the rotation of the rod at the time $t-\tau$.

In this talk, numerical inclusion of exact periodic solutions of Eq.(1) is done through numerical verification method. An inclusion result via constructive implicit function theorem using verified numerical method is shown in Fig.2: From this figure, we have the following conjectures:

Conjeture 1. The number of periodic solutions is finite for a given τ. If we increase τ, the number of periodic solutions increase without limit.

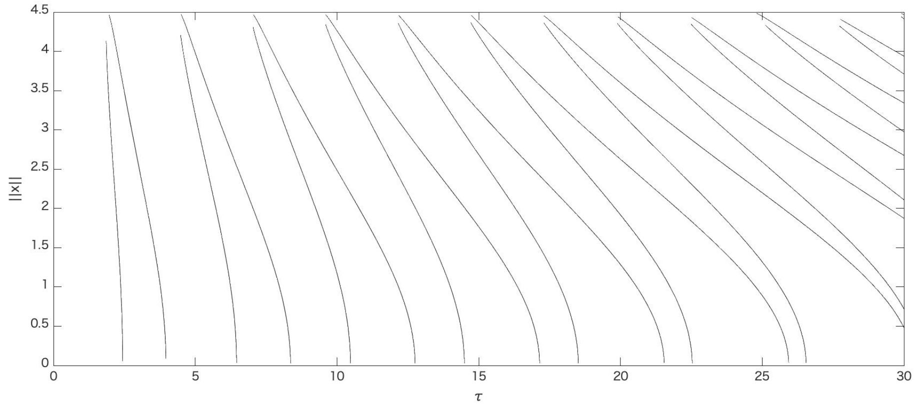

Figure 2: inclusion of a branch consisting of exact periodic solutions

Conjeture 2. Let x be odd symmetric periodic solutions. Then, $\|x\|_2 \leqq 4.51$.

Conjeture 3. Based on this observation, for a given delay $\tau > 0$, a lower bound for the number of periodic solutions is given by floor$(0.32\tau)-1$. Here, for a real r, the functions floor(r) gives the maximum integer less than or equal to r.

As seen in Fig.2, we have proved these conjectures in the range of $0 < \tau \leqq 30$ using verified computations.

Forced Time Delay Duffing Equation

Then, we consider

$$\frac{d^2x(t)}{dt^2} + k\frac{dx(t)}{dt} + \Omega_0^2 x(t) + \gamma x(t)^3 + \alpha(x(t) - x(t-\tau)) = B\cos\omega t \tag{2}$$

with the same parameter choice as before, i.e. $k = 0.1, \Omega_0^2 = 2\gamma = 0.25, \alpha = 0.25$. For fixed B's. we will change ω and observe the behavior of periodic solution of Eq.(2).

I will concern the following conjecture obtained from numerical inclusion results of periodic solutions having the same period as the external force:

Conjeture 4. For $\tau < 0.8$ there is no islands of periodic solutions. For $0.8 < \tau < 2$ the number of islands of periodic solutions are at most one. For $2 < \tau$ the number of islands of periodic solutions are at most that for periodic solutions for the autonomous system with the same delay. For $\tau > 4$ the number of islands of periodic solutions is almost constants regardless B and almost coincides with the number of periodic solutions for the autonomous system with the same delay.

References

[1] S.Oishi: Numerical Verification of Existence and Inclusion of Solutions for Nonlinear Operator Equations, J. Computational and Applied Math., **60**, pp.171-185 (1995)

[2] Rafal Rusinek and Mitura Jerzy Warminski: Time delay Duffing's systems: chaos and chatter control, Meccanica, Volume **49**, Issue 8, pp. 1869-1877 (August 2014).

[3] RK Mitra, S Chatterjee and AK Banik: Limit cycle oscillation and multiple entrainment phenomena in a duffing oscillator under time-delayed displacement feedback, Journal of Vibration and Control 2017, Vol. 23(17) pp.2742-2756.

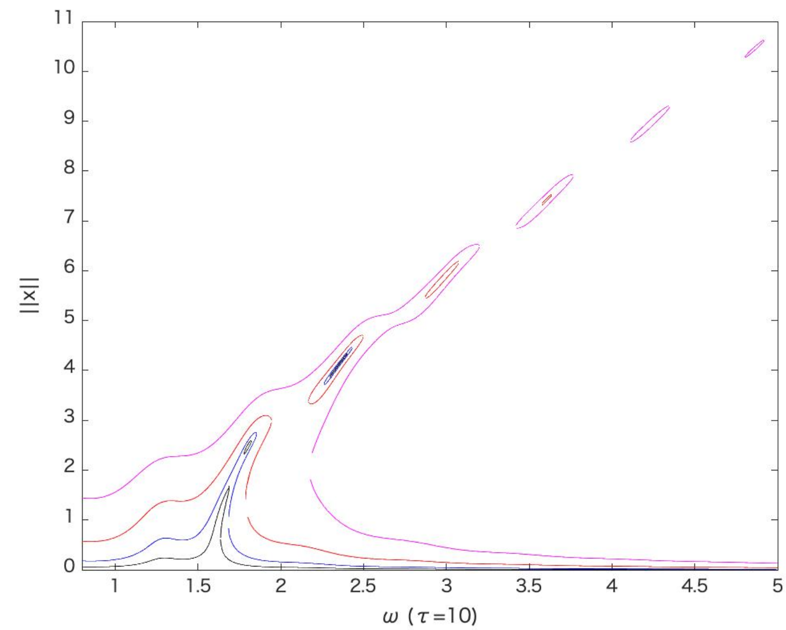

Figure 3: A Verified Periodic solution branches vs. frequencies ω of forcing term ($\tau = 10$ and $B = 0.1(black), 0.3(blue), 1(red), 3(magenta)$.)

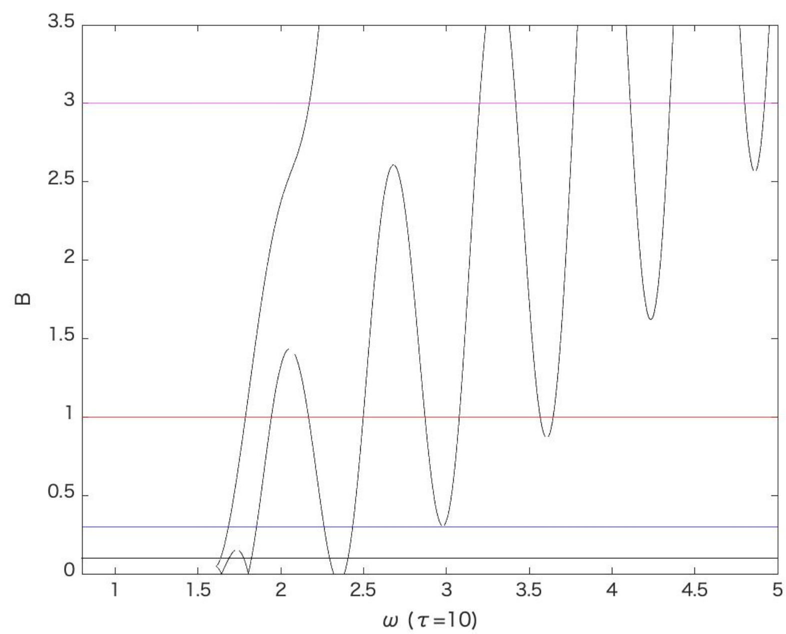

Figure 4: Saddle-Node Bifurcation Set ($\tau = 10$). The black, blue, red and magenta lines show lines of $B = 0.1, B = 0.3$, $B = 1$ and $B = 3$, respectively.