# Quantum symmetry origin of Heisenberg commutation relations: Projective representations of the inhomogneous symplectic group

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A quantum symmetry may be defined as the representation $\rho$ of a group $\mathscr{G}$ that leaves the physically measurable transition probabilities invariant.

$$P(\beta \to \alpha) = |\langle\psi_\beta|\psi_\alpha\rangle|^2 = |\langle\psi_\beta|\rho(g)|\psi\alpha\rangle|^2, \quad |\psi_\delta\rangle \in \mathbf{H}, g \in \mathscr{G}$$

Wigner showed that these representations $\rho$ are always equivalent to a unitary linear or anti-unitary anti-linear repesentation $\nu$. Furthermore, if $\nu(g)$ for $g \in \mathscr{G}$ leaves invariant the transition probabilities, so do also the representations $\nu(\tilde{g})$ where $\tilde{g}$ are elements of a central extension $\widetilde{\mathscr{G}})$ that satisfies the short exact sequence $e \to \mathscr{A} \to \widetilde{\mathscr{G}} \to \mathscr{G} \to e$ for some abelian group $\mathscr{A}$ injected into the center of $\widetilde{\mathscr{G}}$. If $\mathscr{G}$ is not connected, there may be multiple inequivalent central extensions. However, if the group is connected, then only unitary representations are admissable and the central extension is unique and maximal. The quantum symmetries are then fully characterized by the ordinary unitary representations of the central extension of the symmetry group $\mathscr{G}$ with respect to the direct product of the fundamental homotopy group and the second cohomology group, $\mathscr{A} \simeq \pi_1(\mathscr{G}) \otimes H_2(\mathscr{G}, \mathbb{R})$.

For special relativistic quantum mechanics, where the symmetry is the inhomogeneous Lorentz group $IO(1,n)$, $n = 3$, the results are well known; $H^2(IO(1,n), \mathbb{R}) \simeq e$ and so the central extension is the topological central extension $I\overline{O}(1,n)$. $O(1,n)$ is not connected and there are 4 inequivalent topological central extension including the two inequivalent *Pin* group. The cover of the connected Lorentz subgroup $\mathscr{L}(1,3) \subset IO(1,n)$ is uniquely the spin group $Spin(1,3) \simeq S\mathscr{L}(2,\mathbb{C})$. Irreducible unitary representations of the central extension characterize the inertial special relativistic particle states that are labeled by the eigenvalues of the representations of the mass and spin Casimir invariants. This is funda-

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mental to our understanding of special relativistic quantum mechanics and the departure point for quantum field theory [1].

However, the above quantum symmetry formulation makes no mention and gives no insight into the Heisenberg commutation relations that are foundational to quantum mechanics. To understand the quantum symmetry origin of these relations, we study the inhomogeneous symplectic group $ISp(2n)$ that is a symmetry of Hamilton's mechanics on a flat manifold. Using exactly the same methods as above for the connected groups case, we can determine the projective representations of $ISp(2n)$ that are equivalent to the unitary representations of its unique central extension. In this case, $H^2(ISp(2n), \mathbb{R}) \simeq \mathscr{A}(1)$, $\mathscr{A}(m) \simeq (\mathbb{R}^m, +)$ and the central extension is $\mathscr{H}\overline{Sp}(2n) \simeq \overline{Sp}(2n) \otimes_s \mathscr{H}(n)$ where $\mathscr{H}(n)$ is the Weyl-Heisenberg group that is the central extension $e \to \mathscr{A}(1) \to \mathscr{H}(n) \to \mathscr{A}(2n) \to e$. This is the quantum symmetry origin of the the Weyl-Heisenberg group; the inhomogeneous symplectic group is a symmetry of classical Hamiltons mechanics and the corresponding quantum symmetry is given by its projective representations that are equivalent to the ordinary unitary representations of its central extension. The Heisenberg commutation relations are the Hermitian representation of the Lie algebra of the Weyl-Heisenberg normal subgroup of the central extension of the inhomogeneous symplectic group.

We compute the unitary representations of $\mathscr{H}\overline{Sp}(2n)$ using the same Mackey methods as for the inhomogeneous Lorentz group. These theorems originate in the constraint that the semidirect product structure of the group places on the unitary dual. However, as the normal subgroup $\mathscr{H}(n)$ is nonabelian, the manner in which the constraint on the unitary dual determines the unitary irreducible representations is quite different than the inhomogeneous Lorentz group with its abelian normal subgroup. We will show how the unitary irreducible representations of $\mathscr{H}\overline{Sp}(2n)$ are expressed in terms of the ordinary unitary irreducible representations of the cover of the symplectic group, the ordinary unitary irreducible representations of the Weyl-Heisenberg group and the Weil unitary metaplectic representations of the symplectic group [2]. These are equivalent to the faithful projective representations of the inhomogeneous symplectic group.

[1] Weinberg, S. (1995). *The Quantum Theory of Fields, Volume 1*. Cambridge: Cambridge University Press.

[2] Low, S.G. (2014). *Maximal quantum mechanical symmetry: Projective representations of the inhomogeneous symplectic group*. J. Math. Phys. 55, 022105 (2014), arXiv:1207.6787