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The problem of the center for cubic differential systems with invariant algebraic curves

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We consider real cubic differential systems of equations

x'=y+P(x,y),y'=-x+Q(x,y), (1)

where $P(x, y)$ and $Q(x, y)$ have only quadratic and cubic terms. The origin $O(0, 0)$ is a singular point which is a center or a focus (fine focus) for (1). The problem of distinguishing between a center and a focus (the problem of the center) is open for general cubic systems.

It is known that there exists a formal power series $F(x, y) = sumF_k(x, y)$ such that the rate of change of $F(x, y)$ along trajectories of (1) is $dF/dt = \sum_{k=2}^{\infty} L_{k-1}(x^2 + y^2)^k$ where the quantities L_k, $k = 1, 2, \ldots$ are polynomials in the coefficients of system (1) called the Poincaré-Liapunov quantities.

The origin is a center for (1) if and only if $L_k = 0$, $k = 1, 2, .., \infty$. By the Hilbert basis theorem there is a natural number N such that $L_k = 0$ for all k if and only if $L_k = 0$ for all $k < N + 1$. We denote by N_0 the smallest such N.

The problem of the center was solved for quadratic differential systems $N_0 = 3$ and for cubic differential systems with only homogeneous cubic nonlinearities $N_0 = 5$.

The problem we consider in this work is the following:

(i) Find the subclass of cubic differential systems which has a given number M of invariant algebraic curves of respective degrees $d_1, d_2, \ldots, d_M$.

(ii) For this subclass find the least natural number N_0 such that if $L_1 = L_2 = \cdots = L_{N_0} = 0$ then the origin is a center.

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The equations above define the variety of all systems with a center at the origin.

In this talk we discuss this problem in the following cases:

1) $M = 4$ and the curves are lines. We prove that $N_0 = 2$;
2) $M = 3$ and the curves are lines. We prove that $N_0 = 7$;
3) $M = 3$ and the curves are two lines and a conic not passing through the origin. We prove that $N_0 = 4$;
4) $M = 3$ and the curves are two lines and a cubic $a30x^3+a_{21}x^2y+a_{12}xy^2+a_{03}y^3+x^2+y^2 = 0$. We prove that $N_0 = 3$.

We show that the relative position of the invariant algebraic curves in a cubic differential system with a singular point, a fine focus, influence the number of local limit cycles and the number of algebraic limit cycles. In particular, a cubic differential system with a fine focus can have algebraic limit cycles of at most three different degrees.