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Asymptotic freeness of tensor product of unitary matrices

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We consider a family of unitary random matrices $\mathbf{W}_N = (W_1, \dots, W_L)$ in $\mathrm{M}_N(\mathbb{C})^{\otimes K}$ of the form

$$W_\ell = U_\ell^{(N)\otimes K_1} \otimes U_\ell^{(N)t^{\otimes K_2}} \otimes V_\ell^{(N)}, \qquad \ell = 1, \dots, L,$$

where

- $K = K_1 + K_2 + K_3 \geq 1$, where $K_1 \geq 1$ and $K_2, K_3 \geq 0$ are integers.
- $\mathbf{U}_N = (U_1^{(N)}, \dots, U_L^{(N)})$ is a family of $N \times N$ independent Haar unitary matrices ($U_\ell^{(N)t}$ denotes the transpose of $U_\ell^{(N)}$).
- $\mathbf{V}_N = (V_1^{(N)}, \dots, V_L^{(N)})$ is a family of unitary random matrices in $\mathrm{M}_N(\mathbb{C})^{\otimes K_3}$, independent of $\mathbf{U}_N$.

Let $\psi_N : \mathrm{M}_N^{\otimes K}(\mathbb{C}) \to \mathbb{C}$ be a state. Assume that ψ_N or $\mathbf{V}_N$ is invariant by conjugaison by $U \otimes \cdots \otimes U$ for any permutation matrix U and that it satisfies the so-called Mingo-Speicher boundedness condition. Then in the space $(\mathbb{M}_N(\mathbb{C})^{\otimes K}, \psi_N)$, the family $\mathbf{W}_N$ converges in expectation and in probability as $N \to \infty$ to a Haar unitary system.

If $K_2 = 0$, $\Psi_N = \mathbb{E}\Big[\frac{1}{N^K}\mathrm{Tr}^{\otimes K}\Big]$ and the unitary matrices of $\mathbf{V}_N$ are tensor product of N by N matrices, then this result is a consequence of Voiculescu's asymptotic freeness theorem. In that case no boundedness hypothesis is require for $\mathbf{V}_N$. If $K_3 = 0$, the result is a consequence of Mingo-Popa result on the asymptotic freeness of unitary invariant matrices with their transpose.

At the technical level, the result is based on traffic analysis and the following analysis:

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1. a description of the permutation invariant linear forms of $\mathrm{M}_N(\mathbb{C})^{\otimes K}$ in terms of traces of test graphs,

2. a strengthening of the asymptotic traffic independence theorem for the traces of test graphs normalized by Mingo-Speicher bounded.

This is a joint work with Benoit Collins and Pierre-Yves Gaudreau Lamarre.