Une fête de la géométrie, de l'analyse et de la physique
« Conférence en l'honneur de Niky Kamran, à l'occasion de son 60e anniversaire »
10 au 14 juin 2019

A Celebration of Geometry, Analysis and Physics
"Conference honouring Niky Kamran on his 60th birthday"
June 10-14, 2019

Symplectic and variational operators for scalar evolution equations

Mark Fels *

mark.fels@usu.edu

A generalization of the inverse problem in the calculus of variations is the following. Given a scalar evolution equation $u_t = K(t, x, u, u_x, \ldots)$, does there exist a differential operator $\mathscr{E}$ and Lagrangian L such that

$$\mathscr{E}(\Delta) = \mathbf{E}(L) \tag{1}$$

where $\mathbf{E}$ is the Euler-Lagrange operator. A simple example is given by the potential cylindrical KdV equation,

$$tD_x\left(u_t - u_{xxx} - \frac{1}{2}u_x^2 + \frac{u}{2t}\right) = \mathbf{E}\left(-\frac{1}{2}tu_xu_t + \frac{1}{2}tu_xu_{xxx} + \frac{1}{6}tu_x^3\right). \tag{2}$$

Two related questions in the time-independent case is whether there exists a differential operator $\mathscr{D}$ (with conditions) such that

$$\mathscr{D}(K) = \mathbf{E}(L), \qquad \text{or} \qquad K = \mathscr{D} \circ \mathbf{E}(H). \tag{3}$$

In the first case the equation is said to admit a symplectic Hamiltonian structure. In the second case the equation is said to be Hamiltonian. The time dependent formulation of 3 is slightly different.

By considering the variational bicomplex for $u_t = K$ and using the argument in the paper *The Variational Bicomplex for Hyperbolic Second-order Scalar Partial Differential Equations in the Plane* by Anderson and Kamran, I'll show how the symplectic operator problem, and the variational operator problem are the same (in the time dependent or independent case).

Lastly a theorem which maps Hamiltonian operators to variational operators will be used to generate some interesting examples such as 2.

*Department of Mathematics & Statistics, Utah State University, 3900 Old Main Hill, Logan, UT 84322, USA