On the Derrida-Retaux recursive model

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Overview

Model (Derrida and Retaux) :

Let X_0 be random variable taking values in $\{0, 1, 2, ...\}$ (or more generally in $\mathbb{R}_+$). Define

$$X_{n+1} \overset{\text{law}}{=} (X_n^{(1)} + X_n^{(2)} - 1)^+, \qquad \forall n \geq 0,$$

with two independent copies $X_n^{(1)}$, $X_n^{(2)}$ of X_n.

Question :

What can we say about the asymptotic behaviors of X_n as $n \to \infty$?

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Three regimes :

- $X_n \to \infty$ (exponentially fast) : supercritical regime ;
  - Derrida's conjecture on the free energy ;
  - further discussions.
- $X_n \to 0$ (polynomial decay) : critical regime ;
  - Open questions on the behaviors of X_n.
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Outline

Motivation

Pinning model on a hierarchical structure

Derrida and Retaux' model

Results

Nearly supercritical regime

Critical case

Extension and open questions

Proof

Weaker version of Derrida's conjecture

Heuristics in the critical case

Motivations/Related models

- Toy model of a hierarchical renormalization model ;
- Collet, Eckmann, Glaser and Martin (1984) [a spin-glass model] ;
- Aldous and Bandyopadhyay (2005) "Max-type recursive distributional equations" ;
- Goldschmidt and Przykucki (2016) [Parking problem on a tree] ;
- Branching random walk [the survival probability problem].

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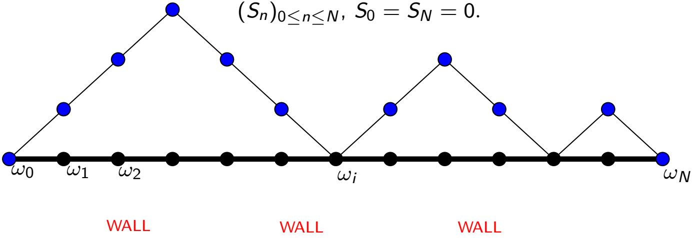
Pining model on $\mathbb{Z}$

$(\omega_i)_{0 \leq i \leq N}$ i.i.d., independent of (S_n);
$(S_n)_{0 \leq n \leq N}$, $S_0 = S_N = 0$.



Pining model on $\mathbb{Z}$

- The measure of polymer of length N :

$$P_{N,\omega}(dS) := \frac{1}{Z_N} \exp\Big(\sum_{i=1}^{N} \omega_i 1_{\{S_i=0\}}\Big),$$

- Z_N is called the partition function [$\omega_i > 0$ (i attractive) ; $\omega_i < 0$ (i repulsive)].
- See Giacomin's book (Random Polymer Models, 2007).

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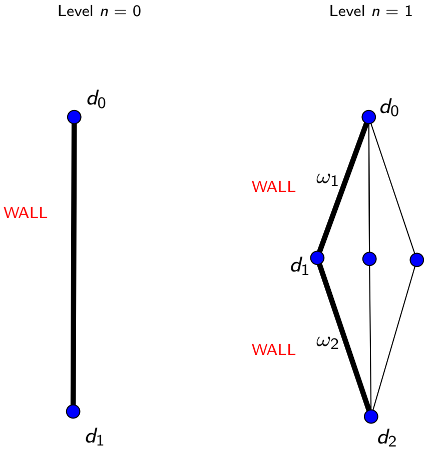
Pinning model on a hierarchical lattice

Derrida, Hakim and Vannimenus (1992)

1. At level 0, there is a unique segment.
2. Fix an integer $B \geq 2$ (for e.g. $B = 3$)
3. Rule : Each segment gives B branches consisting of 2 segments each.

Case $B = 3$



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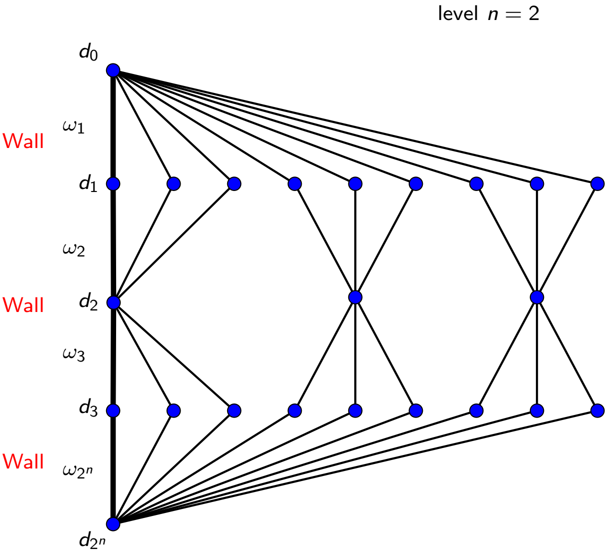

level $n = 2$
d_0
ω_1
Wall
d_1
ω_2
Wall
d_2
ω_3
d_3
Wall
ω_{2^n}
d_{2^n}

Pinning model on a hierarchical lattice

- At level n, each (direct) trajectory $(S_i)_{1\leq i\leq 2^n}$ (from d_0 to d_{2^n}) contains 2^n bonds.
- Choose the uniform measure $P_{B,n}$ on all possible trajectories [simple random walk $(S_i)_{1\leq i\leq 2^n}$ on the hierarchical lattice].
- Let $(\omega_i)_{1\leq i\leq 2^n}$ be i.i.d. and independent of (S_n).
- The partition function

$$Z_n := E_{B,n} \exp\Big(\sum_{i=1}^{2^n} \omega_i 1_{\{S_{i-1}=d_{i-1}, S_i=d_i\}}\Big),$$

where the expectation is only taken with respect to (S_n).

Pinning model on a hierarchical lattice

- Let $N_n :=$ number of trajectories γ from d_0 to d_{2^n} and

$$R_n := \sum_{\gamma:\gamma_0=d_0,\gamma_{2^n}=d_{2^n}} \exp\Big(\sum_{i=1}^{2^n} \omega_i 1_{\{\gamma_{i-1}=d_{i-1},\gamma_i=d_i\}}\Big).$$

- Then $Z_n = \frac{R_n}{N_n}$.
- Easy to see that

$$\begin{aligned} N_{n+1} &= B\,N_n^2, \\ R_{n+1} &= R_n^{(1)} R_n^{(2)} + (B-1)N_n^2, \end{aligned}$$

with two independent copies $R_n^{(1)}$, $R_n^{(2)}$ of R_n.

Pinning model on a hierarchical lattice

- Then
$$Z_{n+1} = \frac{R_{n+1}}{N_{n+1}} = \frac{Z_n^{(1)} Z_n^{(2)} + B - 1}{B},$$
with two independent copies $Z_n^{(1)}$, $Z_n^{(2)}$ of Z_n.
- Stability : $(B, Z) \mapsto (B', Z')$ with $B' := \frac{B}{B-1}, Z' := \frac{Z}{B-1}$.
- See Monthus and Garet (2008), Derrida, Giacomin, Lacoin and Toninelli (2009), Lacoin and Toninelli (2009), Giacomin, Lacoin and Toninelli (2010, 2011), Berger and Toninelli (2013) for the studies of this model [disorder relevance, critical line...]

Pinning model on a hierarchical lattice

- Let $X_n := \log Z_n$. Then

$$\begin{aligned} X_{n+1} &= \log Z_{n+1} \\ &= \log \frac{e^{(X_n^{(1)}+X_n^{(2)})} + B - 1}{B} \\ &\sim X_n^{(1)} + X_n^{(2)}, \qquad \text{if } X_n^{(1)} + X_n^{(2)} \text{ is large.} \end{aligned}$$

- If $X_n \geq -a$ (with $a := -\log(B-1)$), a.s., then $X_{n+1} \geq -a$ a.s.

Derrida and Retaux (2014)'s model

- Fix $a > 0$. For any $n \geq 0$,

$$X_{n+1} \stackrel{\text{law}}{=} \max(X_n^{(1)} + X_n^{(2)}, -a),$$

with two independent copies $X_n^{(1)}$, $X_n^{(2)}$ of X_n.

- Replacing X_n by $X_n + a$ and taking $a = 1$, the recursive equation becomes

$$X_{n+1} \stackrel{\text{law}}{=} (X_n^{(1)} + X_n^{(2)} - 1)^+, \qquad \forall n \geq 0,$$

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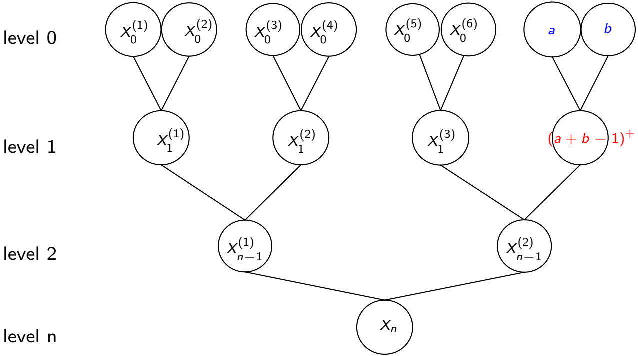

Derrida and Retaux' model

- Free energy : Fact : $F_\infty := \lim_{n\to\infty} \frac{\mathbb{E}(X_n)}{2^n} \in [0,\infty)$ exists.

  Proof : As $X_n \overset{\text{law}}{=} (X_{n-1}^{(1)} + X_{n-1}^{(2)} - 1)^+$,
  $2\mathbb{E}(X_{n-1}) \geq \mathbb{E}(X_n) \geq 2\mathbb{E}(X_{n-1}) - 1$, *implying that*

$$F_\infty := \lim_{n\to\infty} \downarrow \frac{\mathbb{E}(X_n)}{2^n} = \lim_{n\to\infty} \uparrow \frac{\mathbb{E}(X_n) - 1}{2^n}.$$

- Percolation on tree :
  Let

$$X_0 \overset{\text{law}}{=} (1-p)\delta_{\{0\}} + p\delta_{\{\xi\}},$$

with $0 \leq p \leq 1$ and $\xi > 0$ a positive random variable. Define

$$p_c := \sup\{0 \leq p \leq 1 : F_\infty(p) = 0\}.$$

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Question 1 : Value of p_c.

- Example : $\xi \equiv 2$; $p_c = 1/2$?

Question 2 :

- If $p_c < 1$, what is the behavior of $F_\infty(p)$ as $p \downarrow p_c$? [nearly supercritical regime]
- At $p = p_c$, rate of convergence of $X_n \to 0$? [critical regime].

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Value of p_c

Theorem (Collet, Eckman, Glaser and Martin 1984)

Suppose that $\xi \in \{1, 2, ...\}$. Then

$$p_c = \frac{1}{1 + \mathbb{E}((\xi - 1)2^{\xi})}.$$

As example, if $\xi \equiv 2$, then $p_c = \frac{1}{5}$.

Open Problem

Find p_c for a general r.v. $\xi \in \mathbb{R}_+$; or even when $\xi \in \frac{1}{2}\mathbb{N}$?

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Corollary

For any general r.v. $\xi \in \mathbb{R}_+$,

$$p_c > 0 \Longleftrightarrow \mathbb{E}(\xi\, 2^{\xi}) < \infty.$$

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Is there a probabilistic proof on the above $L \log L$-condition ?

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Derrida's conjecture on the free energy

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If $p_c > 0$ (and under some integrability assumptions), then

$$F_\infty(p) = \exp\Big(-\frac{K + o(1)}{(p-p_c)^{1/2}}\Big), \qquad p \downarrow p_c,$$

for some explicit constant $K > 0$.

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Survival probability in branching random walk (Gantert, H. and Shi (2011))

Let $(U(x), x \in \mathbb{T})$ be a real-valued branching random walk. Let $u_* := \lim_{n\to\infty} \frac{1}{n} \max_{|x|=n} U(x)$. Under some integrability assumptions, as $\varepsilon \to 0_+$,

$$\mathbb{P}\Big(\exists\,(x_n)_{n\geq 1} : U(x_n) \geq (u_* - \varepsilon)n,\ \forall n \geq 1\Big) = \exp\Big(-\frac{K' + o(1)}{\sqrt{\varepsilon}}\Big).$$

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Assumption and Classification

Main assumption

ξ takes values in $\{1, 2, ...\}$ and

1. either $\mathbb{E}(\xi^3 2^\xi) < \infty$
2. or $\exists\, \alpha \in (-\infty, 4]$ such that $\mathbb{P}(\xi > x) \approx x^{-\alpha} 2^{-x}, \qquad x \to \infty.$

Consequence of Collet, Eckman, Glaser and Martin 1984 :

$p_c > 0$ iff $\alpha > 2$ (or $\mathbb{E}(\xi^3 2^\xi) < \infty$).

Classification :

1. The supercritical regime : $p > p_c$ ($p_c > 0$ and $p_c = 0$).
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Nearly supercritical regime, case $p_c > 0$: Chen, Dagard, Derrida, H., Lifshits, Shi (2018+)

Theorem 1 (a weaker version of Derrida's conjecture)

If $\mathbb{E}(\xi^3 2^\xi) < \infty$, then

$$F_\infty(p) = \exp\Big(-(p - p_c)^{-\frac{1}{2} + o(1)} \Big), \qquad p \downarrow p_c.$$

Nearly supercritical regime, case $p_c > 0$: Chen, Dagard, Derrida, H., Lifshits, Shi (2018+)

Theorem 2

If $2 < \alpha \leq 4$, then

$$F_\infty(p) = \exp\Big(-(p-p_c)^{-\frac{1}{\alpha-2}+o(1)}\Big), \qquad p \downarrow p_c.$$

Nearly supercritical regime, case $p_c = 0$: H., Shi (2018), Chen, Dagard, Derrida, H., Lifshits, Shi (2018+)

Theorem 3

If $\alpha < 2$, then

$$F_\infty(p) = \exp\Big(- p^{-(1+o(1))/(2-\alpha)}\Big), \qquad p \downarrow 0.$$

Remark :

If $\alpha = 2$ (and if $\mathbb{P}(\xi > x) \sim c\, x^{-2} 2^{-x}$), then

$$F_\infty(p) = \exp\Big(- e^{(c+o(1))/p}\Big), \qquad p \downarrow 0.$$

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Remark on the assumption : the polynomial decay case H., Shi (2018)

Proposition

If

$$\mathbb{P}(\xi > x) \approx \theta^{-x}, \qquad x \to \infty,$$

for some $1 < \theta < 2$, then

$$F_\infty(p) \approx p^{\frac{\log 2}{\log(2/\theta)}}, \qquad p \downarrow 0.$$

Critical case : Chen, Derrida, H., Lifshits, Shi (2018+)

Theorem 4
If $\mathbb{E}(\xi^3 \, 2^\xi) < \infty$ and $p = p_c$, then

$$\mathbb{E}(2^{X_n}) - 1 \approx \frac{1}{n}, \qquad n \to \infty.$$

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Conjectures

If $\mathbb{E}(\xi^3\, 2^\xi) < \infty$ and $p = p_c$, then

- $$\mathbb{E}(2^{X_n}) - 1 \sim \frac{2}{n}, \qquad n \to \infty.$$
- $$\mathbb{E}(X_n) \sim \frac{8}{n^2}, \qquad n \to \infty.$$
- $$\mathbb{P}(X_n \neq 0) \sim \frac{4}{n^2}, \qquad n \to \infty.$$
- Conditionally on $\{X_n \neq 0\}$, $X_n \xrightarrow{(d)} \text{Geometric}(\frac{1}{2})$.

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Derrida-Retaux' Model in the Galton-Watson case

Let ν be an integer-valued r.v. such that $m := \mathbb{E}(\nu) \in (1, \infty)$. Consider the recursive equation

$$X_{n+1} \overset{\text{law}}{=} (\sum_{i=1}^{\nu} X_n^{(i)} - 1)^+,$$

where $X_n^{(1)}, X_n^{(2)}, \ldots,$ are i.i.d. copies of X_n, and independent of ν.

Suppose $X_0 \stackrel{\text{law}}{=} (1-p)\delta_0 + p\delta_\xi$, with $\xi > 0$ a.s. Let

$$F_\infty(p) := \lim_{n\to\infty} \frac{1}{m^n}\mathbb{E}(X_n) \in [0,\infty)$$

and define

$$p_c := \sup\{0 \le p \le 1 : F_\infty(p) = 0\}.$$

Open question

What is the value of p_c in the Galton-Watson case, even if ξ takes integer-values ?

Only known when ν equals an integer a.s. [Collet et al. (1984)].

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Galton-Watson case

Open question

Behavior of the free energy (Derrida's conjecture) when $\nu \neq$ *constant* ?

Proof : elementary observations

- From $X_{n+1} \overset{\text{law}}{=} (\sum_{i=1}^{2} X_n^{(i)} - 1)^+$, we get that

$$2\mathbb{E}(X_n) - 1 \leq \mathbb{E}(X_{n+1}) \leq 2\mathbb{E}(X_n).$$

- Then $F_\infty(p) = \lim \uparrow 2^{-n}(\mathbb{E}(X_n) - 1) = \lim \downarrow 2^{-n}\mathbb{E}(X_n)$.
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Proof of the polynomial decay case

Proposition

If

$$\mathbb{P}(\xi > x) \approx \theta^{-x}, \qquad x \to \infty,$$

for some $1 < \theta < 2$, then

$$F_\infty(p) \approx p^{\frac{\log 2}{\log(2/\theta)}}, \qquad p \to 0.$$

Proof of the polynomial decay : the lower bound of F_∞

- Let $M_n := \max_{1\le i\le 2^n} X_0^{(i)}$. Then $X_n \ge (M_n - n)^+$.
- Extreme value theory implies that

$$M_n \approx \frac{1}{\log\theta}\left(n\log 2 - \log\frac{1}{p}\right).$$

- Then

$$\mathbb{E}(X_n) \ge \mathbb{E}(M_n) - n \approx \left(\frac{\log 2}{\log\theta} - 1\right)n - \frac{\log(1/p)}{\log\theta} > 1,$$

for $n \approx \frac{\log(1/p)}{\log 2/\theta}$.

- Then $F_\infty(p) \approx 2^{-n_0} \ge p^{\frac{\log 2}{\log(2/\theta)}}$ as $p \to 0$.

Proof in the nearly supercritical regime

A weaker version of Derrida's conjecture :

If $\mathbb{E}(\xi^3 2^\xi) < \infty$, then

$$F_\infty(p) = \exp\Big(- (p - p_c)^{-\frac{1}{2}+o(1)}\Big), \qquad p \downarrow p_c.$$

Lower bound : coupling with the critical regime

- Let (Y_n) be a Derrida-Retaux process in the critical regime (i.e. $Y_0 \overset{\text{law}}{=} (1-p_c)\delta_0 + p_c\delta_\xi$).
- Let $\mathbb{T}$ be an infinite binary tree with $Y(x), |x| = 0$ being i.i.d. copies of Y_0.
- Define for any $x \in \mathbb{T}$, $Y(x) := (Y(x^{(1)}) + Y(x^{(2)}) - 1)^+$.
- Let $\mathfrak{e}_n$ be the first lexicographic vertex in the n-th generation of the binary tree $\mathbb{T}$ [then $Y_n \overset{\text{law}}{=} Y(\mathfrak{e}_n)$ for any $n \geq 0$].

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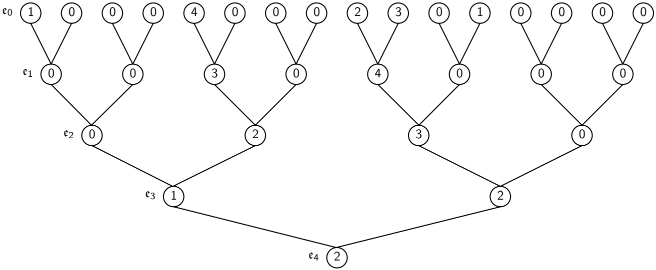

Open paths (in red) in $Y(x), x \in \mathbb{T}$

- For any $x \in \mathbb{T}$, we call $(x_k, 0 \le k \le |x|)$ a path leading to x if x_{k+1} is the (unique) child of x_k for any $0 \le k < |x| - 1$.
- A path is said open if for any vertex x in the path, $Y(x^{(1)}) + Y(x^{(2)}) \ge 1$.
- For $x \in \mathbb{T}$, denote by $N^{\#}(x)$ the number of open paths leading to x. [Below, $N^{\#}(\mathfrak{e}_4) = 6$]

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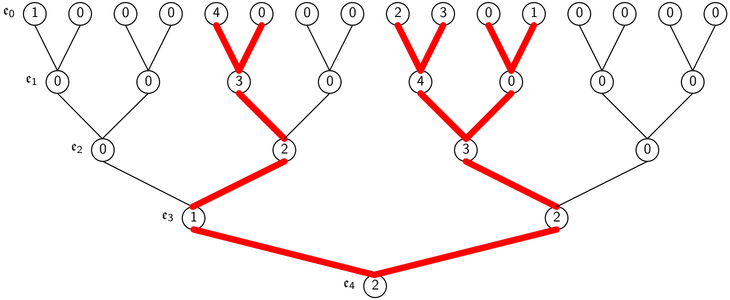

Coupling between supercritical regime and critical regime

- Define (X_0, Y_0) such that $X_0 \overset{\text{law}}{=} (1-p)\delta_0 + p\delta_\xi$ with $p = p_c + \varepsilon$ such that $X_0 \geq Y_0$ a.s. and $\mathbb{P}(X_0 = Y_0 | Y_0 > 0) = 1$.
- Define $X(u), u \in \mathbb{T}$ as for $Y(u), u \in \mathbb{T}$.
- Let $N_n^{(0)}$ be the number of open paths $(x_i)_{0 \leq i \leq n}$ leading to $\mathfrak{e}_n$ such that $Y(x_0) = 0$. Then

$$\mathbb{E}(X_n) \geq c\,\varepsilon\,\mathbb{E}(N_n^0).$$

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Recursive formula for number of open paths

Write $Y_n = Y(\mathfrak{e}_n)$ and $N_n = N^{\#}(\mathfrak{e}_n)$ or $N_n^{(0)}$. Then

$$\mathbb{E}\Big(2^{Y_n}(1+Y_n)N_n\Big) = \mathbb{E}\Big(2^{Y_0}(1+Y_0)N_0\Big) \prod_{k=0}^{n-1} \mathbb{E}(2^{Y_k}).$$

Theorem

We have that

$$\prod_{k=0}^{n-1} \mathbb{E}(2^{Y_k}) \approx n^2.$$

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Number of open paths

- There are some positive constants c, c', c'' such that for all $n \geq 2$, there exists an integer $0 \leq k < cn\,(\log n)^{1/2}$,

$$\mathbb{E}\Big(N_n^{(0)} 1_{\{N_n^{(0)} > c' n^2,\, Y_n = k\}}\Big) \geq \frac{c''}{2^k \log n}.$$

- (Coupling inequality) Recall that $N_n^{(0)}$ denotes the number of open paths $(x_i)_{0 \leq i \leq n}$ leading to $\mathfrak{e}_n$ such that $Y(x_0) = 0$. Then for all $r \geq 0, n, k \geq 1, \ell \leq \varepsilon\, r/2$,

$$\mathbb{E}(X_{n+k+\ell}) \geq 2^{k+\ell-1}\,\varepsilon\,\mathbb{E}\Big(N_n^{(0)} 1_{\{N_n^{(0)} \geq r,\, Y_n = k)\}}\Big).$$

Lower bound of Derrida's conjecture

By the coupling inequality, for all $\ell \leq c\varepsilon n^2$ and for some $k \leq c' n\,(\log n)^{1/2}$,

$$\mathbb{E}(X_{n+k+\ell}) \geq 2^{k+\ell-1}\varepsilon\,\frac{c''}{2^k \log n} \approx \frac{2^\ell \varepsilon}{\log n} > 1,$$

if we choose $\ell \approx \log(1/\varepsilon) + \log\log n$ (and such that $\ell \leq c\varepsilon n^2$). Hence

$$n_0 \leq n + k + \ell \lesssim n(\log n)^{1/2} + \varepsilon n^2$$

with $\varepsilon n^2 \gg \log(1/\varepsilon) + \log\log n$. The optimization says that $n_0 \lesssim \sqrt{(1/\varepsilon)\log(1/\varepsilon)}$. Then

$$F_\infty(p_c + \varepsilon) \geq 2^{-n_0} \geq \exp(-\varepsilon^{-(1+o(1))/2}).$$

Upper bound : a rough estimate

Let $G_n(s) := \mathbb{E}(s^{X_n})$. Using the recurrence equation on the generating functions :

$$G_{n+1}(s) = \frac{1}{s}\, G_n(s)^2 + (1 - \frac{1}{s}) G_n(0)^2 \,.$$

Let $\Delta_n := G_n(2) - G_n^Y(2) \geq 0$. Since $G_n(0) = \mathbb{P}(X_n = 0) \leq \mathbb{P}(Y_n = 0) = G_n^Y(0)$, we have

$$\Delta_{n+1} \leq \frac{1}{2} \left[G_n(2)^2 - (G_n^Y(2))^2 \right] \leq \Delta_n \, G_n(2).$$

Since $G_n(2) = G_n^Y(2) + \Delta_n \leq G_n^Y(2) \mathrm{e}^{\Delta_n}$, we get

$$\Delta_{n+1} \leq \Delta_n \, G_n^Y(2) \, \mathrm{e}^{\Delta_n}.$$

Hence

$$\Delta_{n+1} \leq \Delta_0 \prod_{i=0}^{n} G_i^Y(2) \, \mathrm{e}^{\sum_{i=0}^{n} \Delta_i} \leq c \, \varepsilon \, n^2 \, \mathrm{e}^{\sum_{i=0}^{n} \Delta_i}.$$

Upper bound : a rough estimate

Let $N_1 := \inf\{n \geq 0 : \sum_{i=0}^{n} \Delta_i \geq 1\}$. For all $i < N_1$, $\Delta_{i+1} \lesssim \varepsilon i^2$.
Then for all $n < N_1$,

$$\sum_{i=0}^{n} \Delta_i \lesssim \varepsilon n^3.$$

This implies that $N_1 \geq c\,\varepsilon^{-1/3} =: K$.
Consequently, for any $n < K$,

$$\Delta_{n+1} \leq c\,\varepsilon\,n^2\,\mathrm{e}^{\sum_{i=0}^{n} \Delta_i} \leq c'.$$

Hence $\mathbb{E}(X_K) = \mathbb{E}(Y_K) + \Delta_K \leq 2c'$ and

$$F_\infty(p_c + \varepsilon) \leq \frac{\mathbb{E}(X_K)}{2^K} \leq \mathrm{e}^{-c\,\varepsilon^{-1/3}}.$$

Heuristics in the critical case

Fact

Write $G_n(s) := \mathbb{E}(s^{X_n})$ and $\varepsilon_n := G_n(2) - 1$. Then $\prod_{i=1}^{n}(1+\varepsilon_i) \approx n^2$.

Heuristics

- If $n \to \varepsilon_n$ would be regular, then $\varepsilon_n \sim \frac{2}{n}, \qquad n \to \infty$.
- By iteration, $G_{n+1}(s) = \frac{1}{s} G_n(s)^2 + (1 - \frac{1}{s}) G_n(0)^2$. Set $s = 2$ gives $1 + \varepsilon_{n+1} = \frac{1}{2}(1+\varepsilon_n)^2 + \frac{1}{2} G_n(0)^2$.
- Since $G_n(0) = 1 - \mathbb{P}(X_n \neq 0)$, this implies that

$$\mathbb{P}(X_n \neq 0) \sim (\varepsilon_n - \varepsilon_{n+1}) + \frac{1}{2}\varepsilon_n^2.$$

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Heuristics in the critical case

If $\varepsilon_n - \frac{2}{n} \sim \frac{c}{n^2}$, then

$$\mathbb{P}(X_n \neq 0) \sim \frac{4}{n^2}.$$

Using again $G_{n+1}(s) = \frac{1}{s} G_n(s)^2 + (1 - \frac{1}{s}) G_n(0)^2$ and noticing that $G_n(0)^2 = (1 - \mathbb{P}(X_n \neq 0))^2 \sim 1 - 2\mathbb{P}(X_n \neq 0)$:

$$\frac{G_{n+1}(s) - 1}{\mathbb{P}(X_n \neq 0)} = \frac{1}{s} \frac{G_n(s)^2 - 1}{\mathbb{P}(X_n \neq 0)} - 2(1 - \frac{1}{s}) + o(1).$$

If $\frac{G_n(s)-1}{\mathbb{P}(X_n \neq 0)} \to H(s)$ and $\mathbb{P}(X_{n+1} \neq 0) \sim \mathbb{P}(X_n \neq 0)$, then

$$H(s) = \frac{2}{s} H(s) - 2(1 - \frac{1}{s})$$

giving Yaglom's limit of $\mathcal{L}(X_n | X_n \geq 1)$.

THANK YOU