# Biased random walks among random conductances

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Montréal, Challenges in probability and mathematical physics

The talk is based on joint works with Alessandra Faggionato and Michele Salvi:
"1d Mott variable-range hopping with external field"
"Einstein relation and linear response for biased Mott random walk"
and on joint work with Noam Berger and Jan Nagel:
"The speed of biased random walk among random conductances",
and on joint work with Jan Nagel and Xiaoqin Guo:
"Einstein relation and steady states for the random conductance model".

**Long-term goal:**

Understand the "macroscopic parameters" of random walk among random conductances. For instance: if there is a CLT, how does the variance depend on the law of the conductances?

Consider random walk among random conductances with an additional "external field".

How does the velocity depend on the strength of the external field?

What can one say about steady states, i.e. the environment seen from the particle?

# Outline

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Understand the "macroscopic parameters" of random walk among random conductances. For instance: if there is a CLT, how does the variance depend on the law of the conductances?
Consider random walk among random conductances with an additional "external field".
How does the velocity depend on the strength of the external field?
What can one say about steady states, i.e. the environment seen from the particle?

1. Mott random walk aka Mott variable-range hopping
2. The speed of biased random walk in the random conductance model
3. Back to the Mott walks

Mott random walk is a random walk $(\mathbb{Y}_t)_{t\geq 0}$ in a random environment given by a (stationary, ergodic) point process. More precisely, the environment $\omega$ is a marked simple point process $\{(x_i, E_i)\}_{i\in\mathbb{Z}}$ with law $\mathbb{P}$. The $x_i$ are sites in $\mathbb{R}^d$ and the $E_i$ are energy marks, taking values in an interval $[-A, A]$. The random walk $\mathbb{Y}_t$ has state space $\{x_i\}$ and jumps from a site $x_i$ to any other site $x_k \neq x_i$ with probability rate

$$r_{x_i,x_k}(\omega) := \exp\{-|x_i - x_k| - \beta u(E_i, E_k)\}$$

with $u$ a symmetric function, and $\beta$ the inverse temperature.
$u$ should be such that long-range jumps are likely when energetically favourable.

Recurrence, transience and invariance principles have been investigated by Pietro Caputo, Alessandra Faggionato, Alexandre Gaudillière, Pierre Mathieu, Hermann Schulz-Baldes, Dominique Spehner, Tim Prescott and others.

Under some assumptions on the point process and the law of the energy marks, the rescaled walk converges to a BM with covariance matrix $D(\beta)$ times the identity matrix.

How does $D(\beta)$ depend on $\beta$?

For $d \geq 2$, Mott's law predicted (and two of the above references prove) that the limiting covariance matrix of the random walk is $D(\beta)$ times the identity and $D(\beta)$ decays as $\exp\{-\beta^{\gamma}\}$ for some $\gamma \in (0, 1)$.

Our present results are for the one-dimensional case, i.e. $\{x_i\}_{i\in\mathbb{Z}} \subset \mathbb{R}$. Order the sites $x_i$'s in increasing order, with $x_0 = 0$. Denote by $Z_k := x_{k+1} - x_k$ the distances between subsequent sites in the point process. The environment $\omega$ is now given by a double–sided sequence $(Z_k, E_k)_{k\in\mathbb{Z}}$ of random variables, with $Z_k \in (0, +\infty)$ and $E_k \in \mathbb{R}$ for all $k \in \mathbb{Z}$.

We assume that

(A1) The sequence $(Z_k, E_k)_{k\in\mathbb{Z}}$ is stationary and ergodic with respect to shifts;

(A2) $\mathbb{E}[Z_0]$ is finite.

(A3) For $\mathbb{P}$-a.a. $\omega$ and all $\ell \in \mathbb{Z}\setminus\{0\}$, $\omega \neq \theta_\ell\omega$. Here $\theta_\ell\omega := (Z_{k+\ell}, E_{k+\ell})_{k\in\mathbb{Z}}$.

(A4) There exists $c > 0$ satisfying $\mathbb{P}(Z_0 \geq c) = 1$.

Apply an external field: give the walk a bias to the right. Take $\lambda \in (0,1)$ and let

$$r^{\lambda}_{x_i,x_k}(\omega) := \exp\{-|x_i - x_k| + \lambda(x_k - x_i) + u(E_i, E_k)\}$$

with $u$ a symmetric bounded function.

Note that the embedded discrete Markov chain $(Y_n^\lambda)$ is a RW with long-range conductances given by

$$c_{(x_i,x_j)}(\omega) = \mathrm{e}^{\lambda(x_i+x_j)-|x_j-x_i|+u(E_i,E_j)} = c_{(x_j,x_i)}(\omega) \qquad i \neq j \text{ in } \mathbb{Z}\,.$$

### Theorem

*Alessandra Faggionato, NG, Michele Salvi.*
*For* $\lambda \in (0,1)$, *the Mott walk is transient to the right for* $\mathbb{P}$*–almost all realizations of the environment. Assume* $\mathbb{E}\left[\mathrm{e}^{(1-\lambda)Z_0}\right] < +\infty$. *Then the asymptotic velocity*

$$v_{\mathbb{Y}}(\lambda) := \lim_{t\to\infty} \frac{\mathbb{Y}_t^\lambda}{t}$$

*exists a.s. for* $\mathbb{P}$*–almost all realizations of the environment* $\omega$. *It is deterministic, finite and strictly positive.*

## Remark

*If the random variables $Z_k$ are i.i.d. we have the following dichotomy:* $v_{\mathbb{Y}}(\lambda) > 0$ *if and only if* $\mathbb{E}\left[\mathrm{e}^{(1-\lambda)Z_0}\right] < +\infty$, *otherwise* $v_{\mathbb{Y}}(\lambda) = 0$. *Hence,*

$$\exists \lambda_c \geq 0 : \qquad \begin{cases} v(\lambda) > 0 & \forall \lambda > \lambda_c \\ v(\lambda) = 0 & \forall \lambda < \lambda_c . \end{cases}$$

## Remark

*In particular, there are cases in which the limiting speed $v_{\mathbb{Y}}(\lambda)$ is not continuous in $\lambda$. Can construct examples where the random variables $Z_k$ are i.i.d. such that $v_{\mathbb{Y}}(\lambda)$ is zero for $\lambda \in (0, \lambda_c)$ and strictly positive for $\lambda \in [\lambda_c, 1)$.*

### Question

*Is* $v_{\mathbb{Y}}(\lambda)$ *increasing in* $\lambda$ *?*

### Question

*Does the Einstein relation hold, i.e. do we have*

$$v'(0) = \lim_{\lambda \to 0} v(\lambda)/\lambda = D \qquad ?$$

# The Random Conductance Model

For two neighboring vertices $x$ and $y$ in $\mathbb{Z}^d$ with $d \geq 2$, assign to the edge between $x$ and $y$ a nonnegative *conductance* $\omega(x, y)$. Assume that the conductances are iid and uniformly elliptic, i.e. for some $\delta \in (0, 1)$,

$$1 - \delta \leq \omega(x, y) \leq 1 + \delta. \tag{1}$$

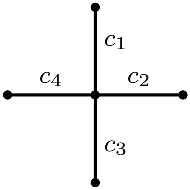
# Biased Random Walks among Random Conductances

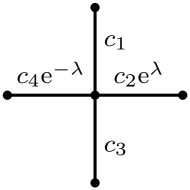
Now consider *biased* random walks among random conductances. Biased RW among the conductances $\omega$ starting at $x_0$ with bias $\lambda \geq 0$, in direction $e_1 = (1, 0, 0, \ldots, 0)$ is the Markov chain $(X_n)_{n \geq 0}$ with law $P^{x_0}_{\omega,\lambda}$, defined by the transition probabilities

$$P^{x_0}_{\omega,\lambda}(X_{n+1} = y | X_n = x) = \frac{\omega(x,y)e^{\lambda(y-x)\cdot e_1}}{\sum_{z \sim x} \omega(x,z)e^{\lambda(z-x)\cdot e_1}}$$

for $x \sim y$.
(We write $x \sim y$ if $x, y$ are neigboring vertices, and we write $w \cdot z$ for the scalar product of two vectors $w, z \in \mathbb{R}^d$). Let $\mathbb{P}_\lambda = \int P_{\omega,\lambda}(\cdot)P(d\omega)$ be the averaged or *annealed law*.

Example: $d = 2, e_1 = (1, 0)$.



### Theorem

*Lian Shen, 2002.*
*$(X_n)$ is transient under $P_{\omega,\lambda}$ for $P$-almost all $\omega$ and the limiting velocity*

$$v(\lambda) = \lim_{n\to\infty} \frac{X_n}{n}$$

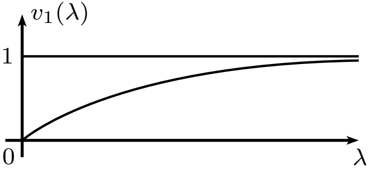
*exists $\mathbb{P}_\lambda$- almost surely and does not depend on $\omega$. Moreover, there is no zero speed regime: let $v_1(\lambda) = v(\lambda) \cdot e_1$. Then, $v_1(\lambda) > 0$ for all $\lambda > 0$.*

# The Monotonicity Question

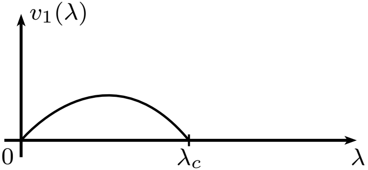
**Question**

*Assume that the conductances are iid and uniformly elliptic.*
*Is $\lambda \to v_1(\lambda)$ increasing?*

For the homogeneous medium, i.e. for constant conductances, $v(\lambda)$ can be computed and $v_1(\lambda)$ looks as follows:

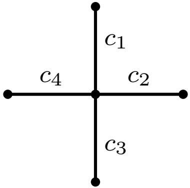


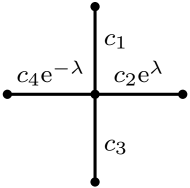
Take supercritical iid bond percolation and condition on the event that the origin is in the infinite cluster. For the speed of the random walk on the infinite percolation cluster, the following picture is conjectured:
For each $p \in (p_c, 1)$ we have, with $v_1(\lambda) = v(\lambda) \cdot e_1$,




(The sharp phase transition in $\lambda$ for positive/zero speed was proved by Alex Fribergh and Alan Hammond).

It is easy to see that for uniformly elliptic conductances, $v_1(\lambda) \to 1$ for $\lambda \to \infty$. A more involved coupling argument due to Gérard Ben Arous, Alex Fribergh and Vladas Sidoravicius yields that $\lambda \to v_1(\lambda)$ is increasing for $\lambda$ large enough.



**Theorem**

*Monotonicity. Noam Berger, NG, Jan Nagel*
*Assume that the conductances are iid and uniformly elliptic. There exists a* $\delta_0 \in (0,1)$, *such that if* $1-\delta_0 \le \omega(x,y) \le 1+\delta_0$ *whenever* $x \sim y$, *then* $v_1$ *is strictly increasing.*

(Low disorder regime).

On the other hand, uniform ellipticity of the conductances does in general not imply monotonicity of the speed.

### Theorem

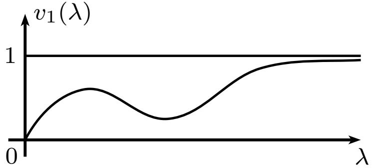
*Non-Monotonicity. Noam Berger, NG, Jan Nagel*
*Assume that the conductances are iid and uniformly elliptic and* $d = 2$*.*
*Define the environment law by*

$$P(\omega(0,e) = 1) = p = 1 - P(\omega(0,e) = \kappa)$$

*for* $p \in (0,1)$ *and* $\kappa > 0$*. Then, for p close enough to 1 and* $\kappa$ *close enough to 0, there exist* $\lambda_1 < \lambda_2$ *such that*

$$v_1(\lambda_1) > v_1(\lambda_2).$$

Hence, the answer to the monotonicity question is "it depends"!
We believe that the picture for the case of the above example is



# About the proof

To prove the monotonicity statement, one step is to show that the derivative of the speed is strictly positive on compact intervals $[0, \lambda_u]$. Use the fact that the derivative can be expressed as the covariance of two processes. For this, we define

$$\widetilde{M}(n) = X_n - \sum_{k=0}^{n-1} E_{\omega,\lambda}^{X_k}[X_1 - X_0],$$
$$N(n) = X_n - n \cdot v(\lambda).$$

We show that under $\mathbb{P}_\lambda$, the $2d$-dimensional process $\frac{1}{\sqrt{n}}(\widetilde{M}(n), N(n))$ converges in distribution to a Gaussian limit $(\widetilde{M}, N)$.
(Main tool: regeneration times).

### Theorem

*Derivative of the speed*
*Assume that the conductances are iid and uniformly elliptic. For any $\lambda > 0$, $v$ is differentiable at $\lambda$ with*

$$v'(\lambda) = \mathrm{Cov}_\lambda(\widetilde{M}, N)e_1.$$

### Remark

*The statement is true for $\lambda = 0$ as well - this is the Einstein relation proved by NG, Jan Nagel and Xiaoqin Guo, AoP 2017. In particular, $\lambda \to v_1(\lambda)$ is a continuous function.*

# Why should the derivative of the speed be a covariance: heuristics

A key ingredient is a change of measure argument, due to Joel Lebowitz and Hermann Rost. Take first $\lambda = 0$. For any $t$, the law of $(X_s)_{0\le s\le t}$ under $\mathbb{P}_\lambda$ is absolutely continuous w. r. t. the law of $(X_s)_{0\le s\le t}$ under $\mathbb{P}_0$ and the Radon-Nikodym density is (roughly) an exponential martingale

$$e^{\lambda M(t) - \frac{\lambda^2}{2}\langle M\rangle(t)}$$

where $M(t)_{t\ge 0}$ is a martingale and $\langle M\rangle(t)$ is the bracket of $M(t)$. In particular,

$$\mathbb{E}_\lambda\left[X_t\right] = \mathbb{E}_0\left[X_t e^{\lambda M(t) - \frac{\lambda^2}{2}\langle M\rangle(t)}\right]$$

$$\mathbb{E}_\lambda\left[X_t\right] = \mathbb{E}_0\left[X_t e^{\lambda M(t) - \frac{\lambda^2}{2}\langle M\rangle(t)}\right]$$

Hence

$$\frac{d}{d\lambda}\,\mathbb{E}_\lambda\left[X_t\right]\Big|_{\lambda=0} = \mathbb{E}_0\left[X_t M(t)\right]$$

and

$$\lim_{t\to\infty}\frac{1}{t}\frac{d}{d\lambda}\,\mathbb{E}_\lambda\left[X_t\right]\Big|_{\lambda=0} = \lim_{t\to\infty}\frac{1}{t}\mathbb{E}_0\left[X_t M(t)\right]$$

Exchanging the order of the limits yields

$$\frac{d}{d\lambda} v(\lambda)|_{\lambda=0} = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}_0 \left[X_t M(t)\right]$$

– which is a covariance, provided that $\left(\frac{X(t)}{\sqrt{t}}\right)_t$ and $\left(\frac{M(t)}{\sqrt{t}}\right)_t$ converge in law.

Now, consider the derivative in $\lambda_0 \neq 0$. Let $\bar{\lambda} = \lambda - \lambda_0$. Then

$$\frac{d}{d\lambda}\, \mathbb{E}_\lambda \left[ (X_t - t \cdot v(\lambda_0)) \right] \big|_{\lambda=\lambda_0}$$

$$= \lim_{\bar{\lambda} \to 0} \frac{1}{\bar{\lambda}} \mathbb{E}_{\lambda_0} \left[ (X_t - t \cdot v(\lambda_0)) \frac{dP_{\omega,\lambda}}{dP_{\omega,\lambda_0}} (X_k; 0 \leq k \leq t) \right]$$

A key step is that

$$\frac{dP_{\omega,\lambda}}{dP_{\omega,\lambda_0}} (X_k; 0 \leq k \leq t) \xrightarrow[\bar{\lambda}\to 0, t\to\infty, \bar{\lambda}^2 t = 1]{d} \exp\left\{ M - \tfrac{1}{2}\langle M \rangle \right\}$$

where $M(t) = \widetilde{M}(t) \cdot e_1$ and $M = \widetilde{M} \cdot e_1$ and we recall that $\frac{1}{\sqrt{t}}(\widetilde{M}(t), N(t))$ converges in distribution to a Gaussian limit $(\widetilde{M}, N)$.

Hence, expect with $\bar\lambda = \lambda - \lambda_0$,

$$
\begin{aligned}
&\lim_{\bar\lambda\to 0} \frac{v(\lambda)-v(\lambda_0)}{\bar\lambda} \\
&= \lim_{t\to\infty,\bar\lambda\to 0,\bar\lambda^2 t=1} \frac{1}{\bar\lambda t}\mathbb{E}_\lambda\left[(X_t - t\cdot v(\lambda_0))\right] \\
&= \lim_{t\to\infty,\bar\lambda\to 0,\bar\lambda^2 t=1} \frac{1}{\bar\lambda t}\mathbb{E}_{\lambda_0}\left[(X_t - t\cdot v(\lambda_0))\frac{dP_{\omega,\lambda}}{dP_{\omega,\lambda_0}}(X_k; 0\le k\le t)\right] \\
&= \lim_{t\to\infty,\bar\lambda\to 0,\bar\lambda^2 t=1} \mathbb{E}_{\lambda_0}\left[\left(\frac{X_t - t\cdot v(\lambda_0)}{\sqrt{t}}\right)\frac{dP_{\omega,\lambda}}{dP_{\omega,\lambda_0}}(X_k; 0\le k\le t)\right] \\
&= \mathbb{E}_{\lambda_0}\left[N\cdot e^{M-\frac{1}{2}\langle M\rangle}\right] = Cov_{\lambda_0}(\widetilde{M}, N)e_1\,.
\end{aligned}
$$

To show the non-monotonicity statement, note that the environment measure with

$$P(\omega(0,e)=1)=p=1-P(\omega(0,e)=\kappa)$$

generates a percolation graph consisting of the edges with conductance 1, connected by $\kappa$-edges.
Choose a bias $\lambda_1$, such that the random walk on the percolation cluster has a positive speed and show

$$v_1(\lambda_1)\geq c_0, \tag{2}$$

for a positive $c_0$ independent of $\kappa$. On the other hand, for a larger bias $\lambda_2$, chosen such that the random walk on the percolation cluster has zero speed, we show

$$v_1(\lambda_2)\leq c_0/2, \tag{3}$$

for $\kappa$ sufficiently small. The combination of these two bounds yields the statement.

Many open questions!
For instance: for **which** uniformly elliptic laws of conductances is the speed increasing in $\lambda$?
In the above example, is the speed increasing in $\kappa$?

# Back to the Mott walks

**Theorem**

*Steady states. Alessandra Faggionato, NG, Michele Salvi*
*Define $\psi$ by $\psi(x_k) = k$. The environment viewed from the discrete-time random walk $(Y_n^\lambda)$, i.e. the process $\left(\theta_{\psi(Y_n^\lambda)}\omega\right)_{n\geq 0}$, admits a unique invariant and ergodic distribution $\mathbb{Q}_\lambda$ which is absolutely continuous w.r.t. $\mathbb{P}$. Moreover, $\mathbb{Q}_\lambda$ and $\mathbb{P}$ are mutually absolutely continuous. Write $\varphi_\lambda$ for the local drift of the random walk $(Y_n^\lambda)$, i.e.*

$$\varphi_\lambda(\omega) := \sum_{k\in\mathbb{Z}} x_k \, p_{0,k}^\lambda(\omega) \, .$$

*The asymptotic velocities $v_Y(\lambda)$ and $v_{\mathbb{Y}}(\lambda)$ can be written as*

$$v_Y(\lambda) = \mathbb{Q}_\lambda\left[\varphi_\lambda\right] \quad \textit{and} \quad v_{\mathbb{Y}}(\lambda) = \frac{v_Y(\lambda)}{\mathbb{Q}_\lambda\left[1/(\sum_{k\in\mathbb{Z}} r_{0,k}^\lambda(\omega))\right]} \, .$$

**Theorem**

*Linear response.*
*Suppose* $\mathbb{E}[e^{pZ_0}] < \infty$ *for some* $p > 2$*. Then, for suitable* $f$*, the map* $\lambda \mapsto \mathbb{Q}_\lambda(f)$ *is differentiable at* $\lambda = 0$*, and the derivative can be characterized as a covariance.*

### Theorem

*We have:*

(i) *If* $\mathbb{E}[\mathrm{e}^{2Z_0}] < \infty$, *then* $v_Y(\lambda)$ *is a continuous functions of* $\lambda$;

(ii) *If* $\mathbb{E}[\mathrm{e}^{pZ_0}] < \infty$ *for some* $p > 2$, *then the Einstein relation holds, i.e.*

$$v_Y'(0) = D_Y \,.$$

# Ingredients of the proof

Random walks on the integers among long range random conductances were already studied by Francis Comets and Serguei Popov, “Ballistic regime for random walks in random environment with unbounded jumps and Knudsen billiards”. Unfortunately the assumptions of their theorem are not satisfied in our case... Still, can use their methods.

In order to prove ballisticity, the strategy is
– Truncate the walk by admitting only jumps of size $\leq \rho$ for some (large) integer $\rho$.
– Define regeneration times for the truncated walk, leading to an invariant measure for the environment seen from the walker.
– Give uniform bounds on the Radon-Nikodym densities of these invariant measures.
– Take the limit $\rho \to \infty$ to get a steady state for the original Mott walk.

For the proof of the Einstein relation, we use an analytical method proposed by Stefano Olla. Key statement is the following.

### Theorem

*Fix* $\lambda_* \in (0,1)$ *and suppose that* $\mathbb{E}\left[\mathrm{e}^{pZ_0}\right] < +\infty$ *for some* $p \geq 2$. *Then, it holds*

$$\sup_{\lambda\in[0,\lambda_*]} \left\| \frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_0} \right\|_{L^p(\mathbb{Q}_0)} < \infty \,. \tag{4}$$

How does (4) imply continuity of the velocity?
Write

$$\begin{aligned} v_Y(\lambda) - v_Y(\lambda_0) &= \mathbb{Q}_\lambda[\varphi_\lambda] - \mathbb{Q}_{\lambda_0}[\varphi_{\lambda_0}] \\ &= \mathbb{Q}_\lambda[\varphi_\lambda - \varphi_{\lambda_0}] + \mathbb{Q}_\lambda[\varphi_{\lambda_0}] - \mathbb{Q}_{\lambda_0}[\varphi_{\lambda_0}] \,. \end{aligned}$$

Show that $\lim_{\lambda\to\lambda_0} \|\varphi_\lambda - \varphi_{\lambda_0}\|_{L^2(\mathbb{Q}_0)} = 0$. Apply Cauchy-Schwarz to get for $\lambda \to \lambda_0$

$$\begin{aligned} \left|\mathbb{Q}_\lambda[\varphi_\lambda - \varphi_{\lambda_0}]\right| &= \left|\mathbb{Q}_0\left[\frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_0}(\varphi_\lambda - \varphi_{\lambda_0})\right]\right| \\ &\le \left\|\frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_0}\right\|_{L^2(\mathbb{Q}_0)} \|\varphi_\lambda - \varphi_{\lambda_0}\|_{L^2(\mathbb{Q}_0)} \to 0 \,. \end{aligned}$$

**Thanks for your attention!**