Quenched invariance principle for random walks among random conductances with stable-like jumps

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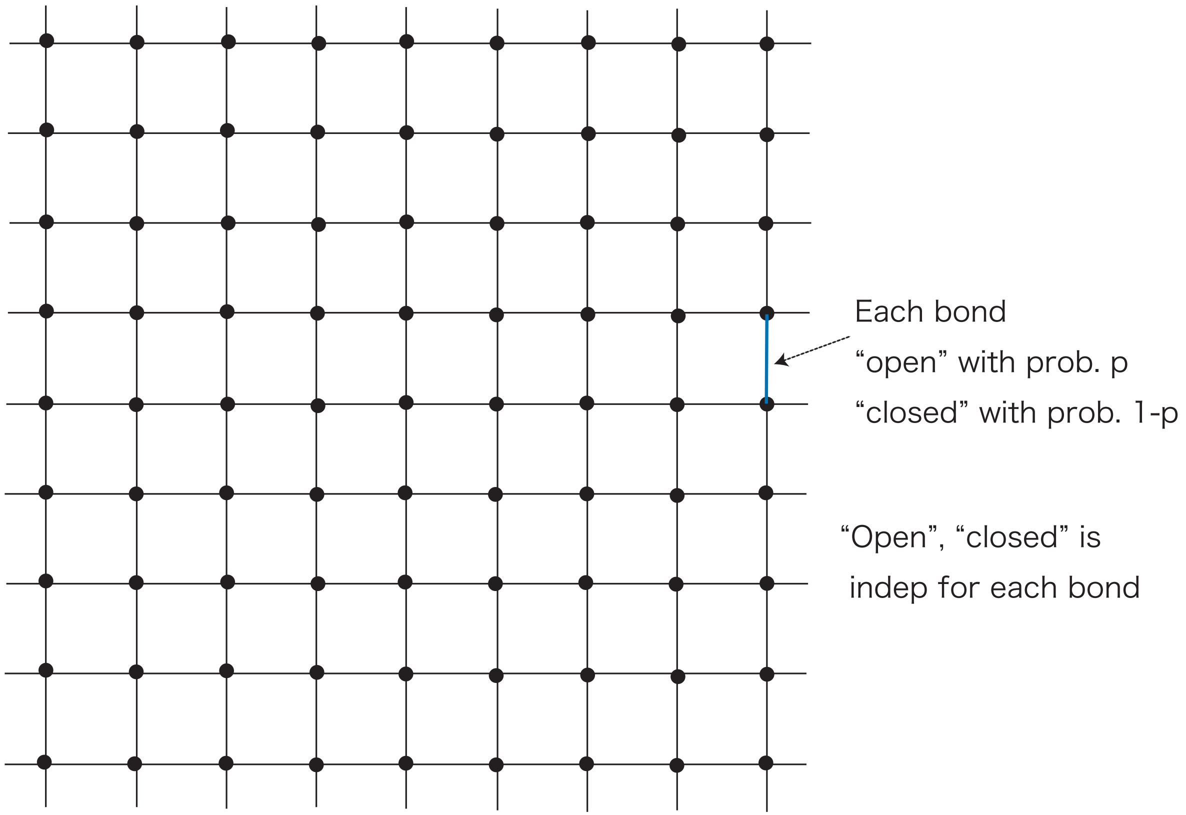
1 Introduction

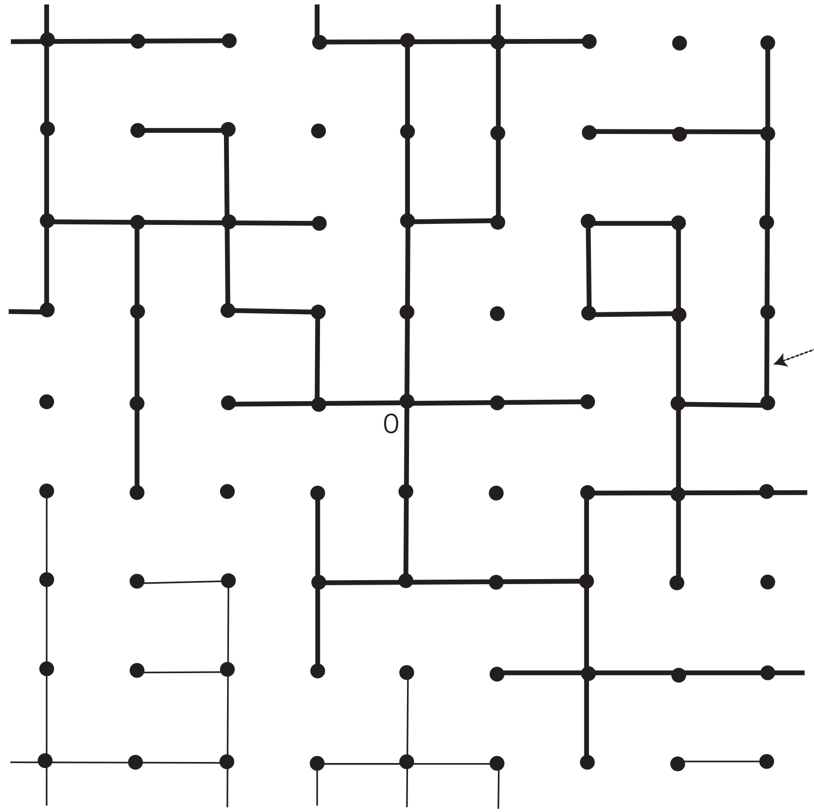
1.1 QIP for nearest neighbor RW

Bond percolation on $\mathbb{Z}^d\,(d \geq 2)$



$\exists p_c \in (0,1)$ s.t. $\exists 1 \infty$-cluster for $p > p_c$, no ∞-cluster for $p < p_c$.

Bond percolation on $\mathbb{Z}^d$ $(d \geq 2)$



Each bond
"open" with prob. p
"closed" with prob. 1-p

"Open", "closed" is
indep for each bond

$\exists p_c \in (0,1)$ s.t. $\exists 1 \infty$-cluster for $p > p_c$, no ∞-cluster for $p < p_c$.

RW on supercriticl percolation on $\mathbb{Z}^d$.

$$\mathcal{E}(f,f) = \frac{1}{2}\sum_{x\sim y} C^\omega_{xy}(f(x)-f(y))^2,$$

$$\mathbb{P}(C^\omega_{xy}=1)=p>p_c,\quad \mathbb{P}(C^\omega_{xy}=0)=1-p.$$

$\{X^\omega_t\}$: corresponding RW.

Theorem (Quenched invariance principle)

Sidoravicius-Sznitman '04, Berger-Biskup '07, Mathieu-Piatnitski. '07

$$n^{-1}X^\omega_{n^2t}\to B_{\sigma t}\ \ \mathbb{P}\text{-a.s. } \omega \text{ for some } \ \sigma>0$$

Remark: i) When $\mathbb{E}C_{xy} < \infty$, "annealed CLT" was obtained in 80's.

(Kipnis-Varadhan '86, De Masi-Ferrari-Goldstein-Wick '89 ($\sigma > 0$))

ii) To prove tightness, HK esitmates play a crucial role. (Barlow '04)

Gaussian HKE holds $\mathbb{P}$-a.s. ω for $t \geq \rho(x,y) \vee \exists U_x$ s.t.

$$c_1 t^{-d/2} \exp\Big(-c_2 \frac{\rho(x,y)^2}{t}\Big) \leq p_t(x,y) \leq \cdots \qquad .$$

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Theorem (Quenched invariance principle II)

(Andres-Deuschel-Slowik '15) $\{C_{xy}\}_{x\sim y}$: positive and stationary ergodic

$$E[(\sum_y C_{xy})^p] < \infty, \quad E[(\sum_y C_{xy}^{-1})^q] < \infty$$

where $1/p + 1/q < 2/d$.

$\Rightarrow \qquad n^{-1}X^{\omega}_{n^2t} \to B_{\sigma t}$ $\mathbb{P}$-a.s. ω for some $\sigma > 0$.

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* (Extensions) Degenerate case: Deuschel-Nguyen-Slowik '18,

Time-dep. case: Biskup-Rodriguez '17, Andres-Chiarini-Deuschel-Slowik '18.

(Q) What if the conductance is no-longer short range?

1.2 HKE for jump processes

Fractional Laplace operator and the α-stable process, $\quad 0 < \alpha < 2$

$$-(-\Delta)^{\alpha/2} f(x) = c_\alpha \int_{\mathbb{R}^d \setminus \{0\}} \frac{(f(x+h) - f(x) - 1_{\{|h|\le 1\}} h \cdot \nabla f(x))}{|h|^{d+\alpha}} dh.$$

$$\frac{\partial}{\partial t} u = -(-\Delta)^{\alpha/2} u$$

with $\lim_{t\to 0} u(t,x) = f(x)$ (f: 'nice' function).

$$\Rightarrow u(t,x) = \int_{\mathbb{R}^d} p_t(x,y) f(y) dy, \qquad \text{where}$$

$$p_t(x,y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}.$$

Integro-differential equation, Polynomial decay when $|x-y|$ large

Corresponding Dirichlet form:

$$\mathcal{E}(f,f) = c_\alpha \int\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}} dxdy.$$

Perturbations of the stable process : systematically studied only recently.

[De Giorgi-Nash-Moser theory]

(Early work) Komatsu '95, Kolokolsov '00, Bass-Levin '02 etc.

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- α-stable-like process (Chen-K '03): $F \subset \mathbb{R}^n$; a d-set.

$$\mathcal{E}(f,f) = \int\int_{F\times F} (u(x)-u(y))^2 J(x,y)\mu(dx)\mu(dy),$$

where $J(x,y) = \frac{c(x,y)}{\rho(x,y)^{d+\alpha}}$, $c(x,y) \in [M^{-1}, M]$ $0<\alpha<2$. Then

$$c_1\left\{t^{-d/\alpha} \wedge \frac{t}{\rho(x,y)^{d+\alpha}}\right\} \le p_t(x,y) \le c_2\left\{t^{-d/\alpha} \wedge \frac{t}{\rho(x,y)^{d+\alpha}}\right\}. \quad (1.1)$$

In fact, $p_t(x,y) \asymp t^{-d/\alpha} \wedge \frac{t}{\rho(x,y)^{d+\alpha}} \Leftrightarrow J(x,y) \asymp \frac{1}{\rho(x,y)^{d+\alpha}}$.

$\Rightarrow$ is easy. Indeed,

$$\mathcal{E}_t(f,g) := (f - P_t f, g)/t \to \mathcal{E}(f,g),$$

$$\mathcal{E}_t(f,g) = \frac{1}{2t} \int\int (f(x) - f(y))(g(x) - g(y)) p_t(x,y) \mu(dx)\mu(dy).$$

Hence $p_t(x,y)/t \to J(x,y)$.

On the other hand, by (HK), $p_t(x,y)/t \asymp 1/\rho(x,y)^{d+\alpha}$ when t small.

So $J(x,y) \asymp \frac{1}{\rho(x,y)^{d+\alpha}}$.

Potential theory on jump processes (non-local operators)

Getting very active both in PDE and probability.

Related recent progress (beyond Lévy proc., incomplete list)

PDE (Integro-diff. eq.) L. Caffarelli, L. Silvestre, S.D. Eidelman, S.D. Ivasyshen, A.N. Kochubei, $\cdots$ (Many others)

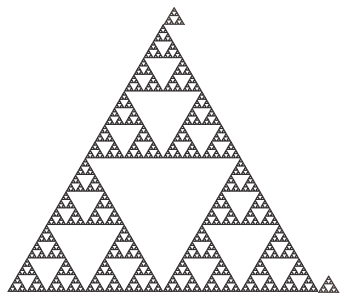
Boundary Harnack/Dirichlet HKE, Subordinated proc K. Bogdan, Z.-Q. Chen, P. Kim, M. Kwaśnicki, R. Song, Z. Vondraček, $\cdots$

Non-sym. jump processes Z.-Q. Chen, X. Zhang, $\cdots$

HKE, Hölder reg., Feller prop. M.T. Barlow, R.F. Bass, B. Dyda, M. Fukushima, A. Grigor'yan, T. Grzywny, E. Hu, J. Hu, T. Jakubowski, M. Kassmann, T. Komatsu, K. Kuwae, A. Mimica, M. Murugan, M. Ryznar, L. Saloff-Coste, R. Schilling, P. Sztonyk, T. Uemura, F.Y. Wang, T. Zheng, $\cdots$

Homogenization M. Arisawa, J.F. Bonder, B. Franke, A. Piatnitski, R. Rhodes, A. Ritorto, A.M. Salort, N. Sandrić, R.W. Schwab, M. Tomisaki, V. Vargas, E. Zhizhina, $\cdots$

Steklov problem, IU R. Bañuelos, T. Kulczycki, $\cdots$



Recent Trilogy: Chen-K-Wang '16-'17

(To appear Mem. AMS, JEMS and J. Math. Pures Appl.)

Stability of HK(ϕ), PHI(ϕ) and EHI on metric measure spaces.

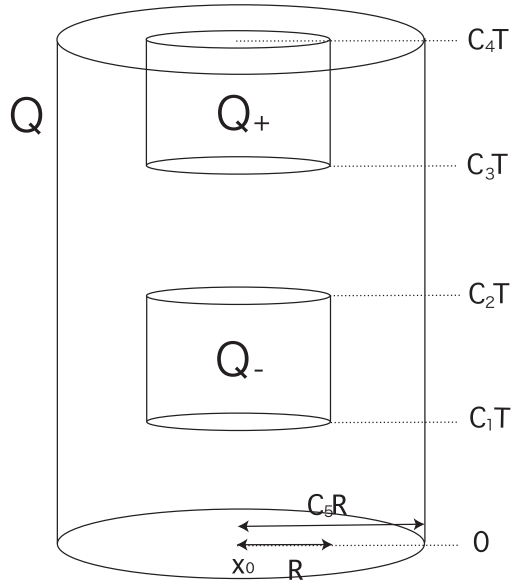
Note: In general, one can define α-stable process even for $2 \leq \alpha < d_w$.

Ex: Sierpinski gasket $d_w = \log 5 / \log 2 > 2$. ($d_w > 2$ for many fractals.)

Difficulty: Lipschitz functions are no longer useful cut-off functions.

Remark: Stability of HKE for d-set also (independently) proved by

Murugan - Saloff-Coste (graph case) and Grigor'yan-Hu-Hu.



• PHI(ϕ) (**parabolic Harnack inequality**):

$\exists C_1 > 0$ s.t. $\forall u = u(t,x) \geq 0$ on $[0,\infty) \times F$ caloric in Q,

$T = \phi(R)$, then $\sup_{Q_-} u \leq C_1 \inf_{Q_+} u$.

• EHI (**elliptic Harnack inequality**): $\exists C_1 > 0$ s.t. $\forall h = h(x) \geq 0$ on F harmonic in $B(x_0, 2R)$, then $\sup_{B(x_0,R)} h \leq C_2 \inf_{B(x_0,R)} h$.

2 Quenched invariance principle

Setting $F := \mathbb{R}_+^{d_1} \times \mathbb{R}^{d_2}$ for $d_1, d_2 \in \mathbb{N} \cup \{0\}$, and let $\mathbb{L} := \mathbb{Z}_+^{d_1} \times \mathbb{Z}^{d_2}$.

$d; = d_1 + d_2$, $\quad w_{x,y}(\omega) = w_{y,x}(\omega) \geq 0$, $x \neq y \in \mathbb{L}$: r.v. Define

$$L_{\mathbb{L}}^{\omega} f(x) = \sum_{y \in \mathbb{L}} (f(y) - f(x)) \frac{w_{x,y}(\omega)}{|x-y|^{d+\alpha}}, \quad x \in \mathbb{L}, \tag{2.1}$$

where $\alpha \in (0,2)$. (We write $w_{x,x}(\omega) = w_{x,x}^{-1}(\omega) = 0$.)

$\{X_t^{\omega}\}_{t \geq 0}$: corresponding Markov process.

$X_t^{(n),\omega} := n^{-1} X_{n^{\alpha}t}^{\omega}$ on $V_n = n^{-1}\mathbb{L}$.

$\mathbb{P}_x^{(n),\omega}$: law of $X_\cdot^{(n),\omega}$ with initial point $x \in V_n$.

Theorem Let $d > 2 - 2\alpha + 21_{\{\alpha\in[1,2)\}}$. $\{w_{x,y}\}_{x,y}$: indep., $\mathbb{E}w_{x,y} = 1$,

$$p_0 := \sup_{x,y\in\mathbb{L},x\neq y} \mathbb{P}\big(w_{x,y} = 0\big) < 2^{-4}, \tag{2.2}$$

$$\sup_{x,y\in\mathbb{L}} \mathbb{E}[w_{x,y}^p] + \mathbb{E}[w_{x,y}^{-q} 1_{\{w_{x,y}\neq 0\}}] < \infty, \tag{2.3}$$

for $p, q \in \mathbb{Z}_+$ with

$$p > \max\Big\{\frac{2(d+2)}{d}, \frac{d+1}{1-\alpha+1_{\{\alpha\in[1,2)\}}}\Big\}, \quad q > \frac{2(d+2)}{d}. \tag{2.4}$$

Then the quenched invariance principle holds, i.e.

$$X_t^{(n),\omega} := n^{-1} X^\omega_{n^\alpha t} \to Y, \quad \mathbb{P}\text{-a.s. } \omega, \tag{2.5}$$

where Y is a sym. α-stable process on F with jumping meas. $|z|^{-d-\alpha}\, dz$.

Example The following example satisfies (2.2) and (2.3): for $x, y \in \mathbb{Z}^d$,

$$\mathbb{P}(w_{x,y} = 0) = 2^{-5}$$

$$\mathbb{P}(w_{x,y} = |x-y|^{\varepsilon}) = (3|x-y|^{2p\varepsilon})^{-1},$$

$$\mathbb{P}(w_{x,y} = |x-y|^{-\delta}) = (3|x-y|^{2q\delta})^{-1},$$

$$\mathbb{P}(w_{x,y} = g(x,y)) = 1 - (3|x-y|^{2p\varepsilon})^{-1} - (3|x-y|^{2q\delta})^{-1} - 2^{-5},$$

where $\varepsilon, \delta > 0$ and $g(x,y)$ is chosen so that $\mathbb{E}w_{x,y} = 1$.

(Note: $c^{-1} \le g(x,y) \le c$ for some $c > 1$.)

Remark:

i) The equivalance of HKE requires $J(x,y) \asymp \frac{1}{\rho(x,y)^{d+\alpha}}$ (pointwize).

Yet, we only need long time behavior, hence requires weak conditions.

ii) Harnack ineqialities do **NOT** hold in long range random media

(except for the uniform elliptic case).

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Yet, we only need long time behavior, hence requires weak conditions.

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Our approach: Develop and use probabilistic potential theory.

Advantage: Can be used without translation inv.

(Method not restricted to $\mathbb{Z}^d$.)

Disadvantage: Need some 'strong mixing' condition for the media

to apply B-C lemma. (Cannot work on stat. ergo. media in general.)

3 Quenched HKE (On-going)

Theorem Suppose that $d > 4 - 2\alpha$, (2.2) and (2.3) for $p, q \in \mathbb{Z}_+$ with $p > \max\left\{\frac{d+1+\theta_0}{d\theta_0}, \frac{d+1}{2\theta_0(2-\alpha)}\right\}$ and $q > \frac{d+1+\theta_0}{d\theta_0}$, where $\theta_0 := \alpha/(2d+\alpha)$.

Then, $\mathbb{P}$-a.s. $\omega \in \Omega$, for $\forall x \in \mathbb{L}$, $\exists R_x(\omega) \geq 1$ s.t.

$\forall R > R_x(\omega)$ and $\forall t > 0$ with $t \geq (|x-y| \vee R_x(\omega))^{\theta_1 \alpha}$,

$$C_1\Big(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\Big) \leq p^\omega(t,x,y) \leq C_2\Big(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\Big),$$

where $C_1, C_2 > 0$ and $\theta_1 \in (0, \alpha/(2d+\alpha))$ (non-random).

[**Local limit theorem for VSRW**]

For $a > 0$, let $k_{a,t}(x) := k_{a,t}(0,x)$ be the HK corresponding to symmetric α-stable processes with Lévy measure $a|z|^{-d-\alpha}\,dz$.

Theorem Under the setting of the previous theorem, assume further that $\mathbb{E}w_{x,y} = a$ for all $x \neq y \in \mathbb{L}$. Then, for any $T_2 > T_1 > 0$ and $k > 1$,

$$\lim_{n\to\infty} \sup_{|x|\le k} \sup_{t\in[T_1,T_2]} |n^d p^\omega(n^\alpha t, 0, [nx]) - k_{a,t}(x)| = 0,$$

where $[x] = ([x_1], \cdots, [x_d])$ for any $x = (x_1, \cdots, x_d) \in \mathbb{R}_+^{d_1} \times \mathbb{R}^{d_2}$.

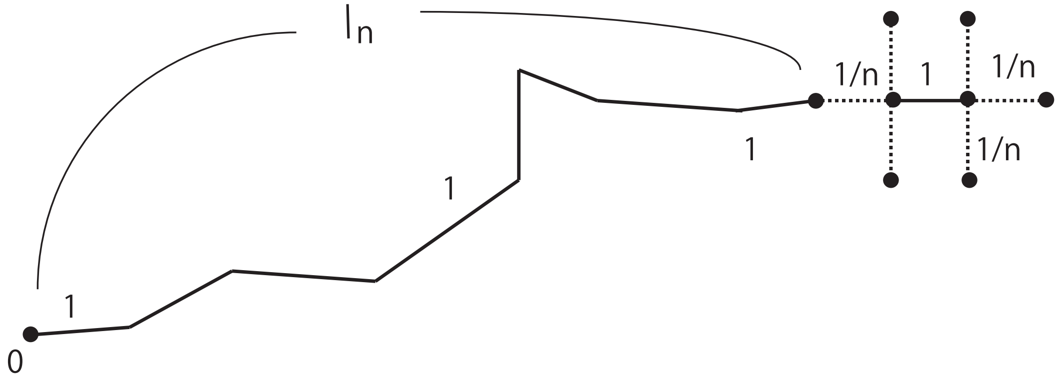
Traps **Moment cond. of $w_{x,y}^{-1}$ is needed.**

Theorem [n.n. case] (Berger-Biskup-Hoffman-Kozma '08)

$\forall \{\lambda_n\}_{n\in\mathbb{N}}$, $\lambda_n \nearrow \infty$, $\exists$ i.i.d. law $\mathbb{P}$ with $\mathbb{P}(0 < \mu_e \leq 1) = 1$ s.t.

$$P_\omega^{2n}(0,0) \geq C_1(\omega) n^{-2} \lambda_n^{-1} \quad \text{for } \; d \geq 5$$

along a subsequence that does not depend on ω.



Jump case, similar example can be made, but far from 'optimal'!

Open prob. Find the optimal p, q. Any trap produced by big jumps?

Long range percolation

$\{C_{xy} : x, y \in \mathbb{Z}^d, x \neq y\}$: indep. Bernoulli r.v.

$$\mathbb{P}(C_{xy} = 1) = \frac{A}{|x-y|^{d+\alpha}}, \quad \mathbb{P}(C_{xy} = 0) = 1 - \frac{A}{|x-y|^{d+\alpha}},$$

where $A > 0, \alpha > 0$.

Crawford-Sly '13: QIP to the α-stable proc. when $0 < \alpha < 1$

(using weaker top.). $1 \leq \alpha < 2$ still open.

— Unfortunately we cannot include this. (Too singular.)

On-going work (Z.Q. Chen, X. Chen, K, J. Wang)

$$\mathcal{E}_\omega(f,f) = \int\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(f(x)-f(y))^2 C(x,y,\omega)}{|x-y|^{d+\alpha}} dxdy.$$

where $\lambda^{-1} \le C(x,y,\omega) = C(y,x,\omega) \le \lambda$, stat. ergo. Then,

$$\varepsilon X^\omega_{t/\varepsilon^\alpha} \to Y, \quad \mathbb{P}\text{-a.s. } \omega,$$

where Y is a sym. α-stable process on $\mathbb{R}^d$

with jumping meas. $\mathbb{E}[C(0,z,\cdot)] \cdot |z|^{-d-\alpha}\, dz.$

(Related work): Kassmann, Piatnitski and Zhizhina. '18

4 Key propositions for QIP

Proposition A $\exists \theta \in (0,1)$, $\exists R_0(\omega) > 0$ s.t. $\forall R \geq R_0(\omega)$, $\forall x_0 \in B(0,R)$, $R^\theta \leq \forall r \leq R$, $\forall t_0 \geq 0$ and $\forall q \geq 0$: parabolic on $Q(t_0, x_0, 2r)$

$$|q(s,x) - q(t,y)| \leq C_1 \|q\|_{\infty,r} \left(\frac{|t-s|^{1/\alpha} + \rho(x,y)}{r} \right)^\beta, \tag{4.1}$$

holds $\forall (s,x),(t,y) \in Q(t_0,x_0,r) := (t_0, t_0 + C_0 r^\alpha) \times B(x_0, r)$ with $(C_0^{-1}|s-t|)^{1/\alpha} + \rho(x,y) \geq 2r^\theta$, where

$$\|q\|_{\infty,r} = \sup_{(s,x)\in[t_0,t_0+C_0(2r)^\alpha]\times\mathbb{L}} q(s,x).$$

Deterministic statements on a graph (V, E) with dist. $\rho(\cdot,\cdot)$.

* From now on, we assume $w_{x,y} > 0$ for simplicity.

(Generalization to (2.2) not hard.)

Assumption (Exi.)

$\exists \theta \in (0,1)$, $\exists o \in V$, $\exists R_0 \geq 1$ *s.t.* $R_0 < \forall R$ *and* $R^{\theta^3}/2 \leq \forall r \leq 2R$,

$$\sup_{x \in B(0,6R)} \sum_{y \in V : \rho(x,y) \leq r} \frac{w_{x,y}}{\rho(x,y)^{d+\alpha-2}} \leq C_1 r^{2-\alpha},$$

$$\sup_{x \in B(0,6R)} \sum_{y \in V : \rho(x,y) \leq c_0 r} w_{x,y}^{-1} \leq C_1 r^d,$$

$$\sup_{x \in B(0,6R)} \sum_{y \in V : \rho(x,y) > r} \frac{w_{x,y}}{\rho(x,y)^{d+\alpha}} \leq C_1 r^{-\alpha}.$$

Proposition B

Under **(Exi.)**, $\forall\theta \in (\theta, 1)$, $\exists R_1 \geq 1$ s.t. $R_1 < \forall R$ and $R^{\theta^2} \leq \forall r \leq R$,

$$\sup_{x\in B(0,2R)} \mathbf{P}_x\left(\tau_{B(x,r)} \leq C_0 r^{\alpha}\right) \leq \frac{1}{4}, \tag{4.2}$$

$$C_2 r^{\alpha} \leq \inf_{x\in B(0,2R)} \mathbf{E}_x\left[\tau_{B(x,r)}\right] \leq \sup_{x\in B(0,2R)} \mathbf{E}_x\left[\tau_{B(x,r)}\right] \leq C_1 r^{\alpha}.$$

Proposition C (Krylov-type estimate)

Under **(Exi.)**, $\exists R_1 \geq 1$ s.t. $R_1 < \forall R < r_G$, $2R^{\theta^2} \leq \forall r \leq R$,

$\forall x \in B(0, 2R)$, $t \geq 0$ and $A \subseteq Q(t, x, r/2)$ with $\frac{\nu(A)}{\nu(Q(t,x,r/2))} \geq \frac{1}{2}$,

$$\mathbf{P}_{(t,x)}(\sigma_A < \tau_{Q(t,x,r)}) \geq C_1, \tag{4.3}$$

where $d\nu = ds \times d\mu$.

Structure of the proof of main theorem

- Tightness (Prop B, (4.2)): Bass-Nash + Trunctation and 'localization'.

Compare τ with that of the truncate-localized process.

- Prop C $\Rightarrow$ Prop A.

- Mosco conv. of $X_\cdot^{(n)}$ to Y $\oplus$ Prop A $\Rightarrow$ f.d.d. conv.

Sufficient cond. for **(Exi.)**: 'strong mixing' condition such as

$$p_1(r,R,\varepsilon) := \mathbf{P}\Big(\Big|\sum_{x,y\in V_0:\rho(0,x)\le R,\rho(x,y)\le r}(w_{x,y}-1)\Big| > \varepsilon r^d R^d\Big),$$

$$\sum_{R=1}^{\infty}\sum_{r=1}^{R} p_1(r,R,\varepsilon_0) < \infty.$$

Can be verified under moment condition in the theorem (by B-C).

Lemma D (Lévy system formula) $f : \mathbb{R}_+ \times V \times V \to \mathbb{R}_+$ that vanishes along the diagonal. Then, $\forall t \geq 0$, $\forall T$: stopping time,

$$\mathbb{E}^x \left[\sum_{s \leq T} f(s, Y_{s-}, Y_s) \right] = \mathbb{E}^x \left[\int_0^T \left(\sum_V f(s, Y_s, y) J(Y_s, y) \right) ds \right].$$

Sketch of proof of Prop C. Let $Q_r := Q(t, x, r)$.

W.l.o.g. we may assume $\mathbf{P}_{(t,x)}(\sigma_A < \tau_{Q_r}) \leq 1/4 \cdots (*)$.

Step 1

$$\mathbf{P}_{(t,x)}(\sigma_A < \tau_{Q(t,x,r)}) \geq \frac{c_1}{r^{d+\alpha}} \Big(\inf_{z \in B(x_0, r)} \int_0^{c_2 r^\alpha} \sum_{u \in A_s} w_{z,u} \, ds \Big), \quad (4.4)$$

where $A_s = \{y \in V : (s, y) \in A\}$.

Let $T = \sigma_A \wedge \tau_{Q_r}$. According to Lemma D,

$$\mathbf{P}_{(t,x)}(\sigma_A < \tau_{Q_r}) \geq \mathbf{E}_{(t,x)}\left(\sum_{s \leq T} 1_{\{X_s \neq X_{s-}, X_s \in A_s\}}\right)$$

$$= \mathbf{E}_{(t,x)}\left[\int_0^T \sum_{u \in A_s} \frac{w_{X_s,u}}{\rho(X_s,u)^{d+\alpha}}\, ds\right]$$

$$\geq \mathbf{E}_{(t,x)}\left[\int_0^{C_0(r/2)^\alpha} \sum_{u \in A_s} \frac{w_{X_s,u}}{\rho(X_s,u)^{d+\alpha}}\, ds : T \geq C_0(r/2)^\alpha\right]$$

$$\geq c_1 r^{-d-\alpha}\left(\inf_{z \in B(x,r)} \int_0^{C_0(r/2)^\alpha} \sum_{u \in A_s} w_{z,u}\, ds\right) \mathbf{P}_{(t,x)}(T \geq C_0(r/2)^\alpha).$$

Note: $\rho(u,z) \leq 2r$ for every $u, z \in B(x,r)$.

Using Prop B, (4.2) and $(*)$,

$$\begin{aligned}\mathbf{P}_{(t,x)}\big(T \geq C_0(r/2)^\alpha\big) &= \mathbf{P}_{(t,x)}\big(\sigma_A \wedge \tau_{Q_r} \geq C_0(r/2)^\alpha\big) \\ &\geq 1 - \mathbf{P}_{(t,x)}\big(\sigma_A \leq \tau_{Q_r}\big) - \mathbf{P}_x\big(\tau_{B(x,r)} \leq C_0(r/2)^\alpha\big) \\ &\geq 1 - \frac{1}{4} - \frac{1}{4} \geq \frac{1}{2},\end{aligned}$$

hence (4.4) is obtained.

STEP 2

$$\inf_{z\in B(x_0,r)} \int_0^{c_2 r^\alpha} \sum_{u\in A_s} w_{z,u}\, ds \geq c_2 r^{d+\alpha}. \tag{4.5}$$

$$\begin{aligned}
\nu(A) &= \int_0^{C_0(r/2)^\alpha} \sum_{u\in A_s} 1\,ds \\
&\le c_3 r^{\alpha/2}\Big(\sum_{u\in B(x,r)} w_{z,u}^{-1}\Big)^{1/2}\Big(\int_0^{C_0(r/2)^\alpha}\sum_{u\in A_s} w_{z,u}\,ds\Big)^{1/2} \\
&\le c_3 r^{\alpha/2}\Big(\sup_{z\in B(0,3R)}\sum_{u\in V:\rho(u,z)\le 2r} w_{z,u}^{-1}\Big)^{1/2}\Big(\int_0^{C_0(r/2)^\alpha}\sum_{u\in A_s} w_{z,u}\,ds\Big)^{1/2} \\
&\le c_4 r^{(d+\alpha)/2}\Big(\int_0^{C_0(r/2)^\alpha}\sum_{u\in A_s} w_{z,u}\,ds\Big)^{1/2} \quad \text{by } \textbf{(Exi.)}.
\end{aligned}$$

Since $\nu(A) \ge \frac{\nu(Q(t,x,r/2))}{2} \ge c_5 r^{d+\alpha}$, we obtain (4.5).

Thank you!