Hydrodynamic limit of particle systems on resistance spaces

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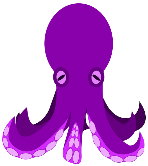

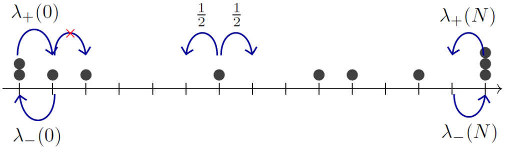

Hydrodynamic program on singular spaces

Goal: Rigorous derivation of fluid equations from interacting particle systems on singular spaces, such as fractals.
Microscopic model: the weakly asymmetric exclusion process on an infinite weighted graph
Macroscopic PDE: a nonlinear heat equation in the diffusive scaling limit

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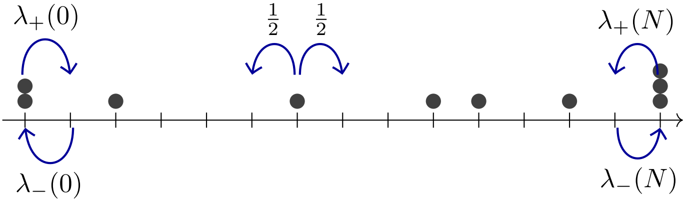
The entire program is presented in 4 parts. **(focus of this talk)**
Parts 3 and 4 are joint works with Michael Hinz (Bielefeld) and Alexander Teplyaev (UConn).

1. **The moving particle lemma for the exclusion process on a finite weighted graph**
   (the analog of Thomson's inequality for random walks—a Sobolev embedding theorem)
   C. '17, `arXiv:1606.01577`. *Electron. Commun. Probab.* **22** (2017), paper no. 47.
2. **Local ergodic theorem for the exclusion process on strongly recurrent graphs**
   C. '17, `arXiv:1705.10290`.
3. **Semilinear evolution equations on resistance spaces**
   C.–Hinz–Teplyaev '18+
4. Hydrodynamic limit (LLN, LDP) of the exclusion process on the Sierpinski gasket
   C.–Hinz–Teplyaev '18+

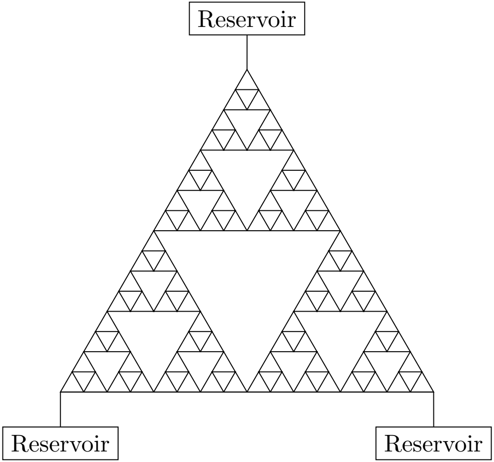
These results are summarized in the review
C.–Hinz–Teplyaev '17, `arXiv:1702.03376`. Appears in the proceedings for the conference "SPDEs and Related Fields" in honor of Michael Röckner's 60th birthday (2018)

The boundary-driven exclusion process

- Particles are indistinguishable.
- In the bulk, $(0, N) \cap \mathbb{Z}$, particles undergo exclusion dynamics. (Random walks subject to the exclusion constraint: no two particles can occupy the same vertex.)
- At each boundary vertex $y \in \{0, N\}$ ("reservoir"), particles can be injected into or extracted from the bulk at resp. rate $\lambda_+(y)$ and $\lambda_-(y)$.
  - If $\lambda_+(y) = \lambda_-(y)$ for all y: system reaches equilibrium.
  - Otherwise: system is out of equilbrium (a mean density gradient develops between a "hot" reservoir and a "cold" one).



Boundary-driven exclusion process on the Sierpinski gasket (SG)



Exclusion process on a weighted graph

Let $G = (V, E)$ be a connected graph endowed with conductances $\mathbf{c} = (c_{xy})_{xy \in E}$.

The symmetric exclusion process on $(G, \mathbf{c})$ is a Markov chain on $\{0,1\}^V$ with generator

$$(\mathcal{L}^{\mathrm{EX}}_{(G,\mathbf{c})} f)(\eta) = \sum_{xy \in E} c_{xy} (\nabla_{xy} f)(\eta). \quad f : \{0,1\}^V \to \mathbb{R},$$

where $(\nabla_{xy} f)(\eta) := f(\eta^{xy}) - f(\eta)$ and $(\eta^{xy})(z) = \begin{cases} \eta(y), & \text{if } z = x, \\ \eta(x), & \text{if } z = y, \\ \eta(z), & \text{otherwise.} \end{cases}$

Properties:

1. Total particle number is conserved in the process.
2. Each product Bernoulli measure ν_α, $\alpha \in [0,1]$, with marginal $\nu_\alpha\{\eta : \eta(x) = 1\} = \alpha$ for each $x \in V$, is an invariant measure for this process.

Dirichlet energy: $\mathcal{E}^{\mathrm{EX}}_{(G,\mathbf{c}),\nu_\alpha}(f) = \dfrac{1}{2} \displaystyle\sum_{zw \in E} c_{zw} \int_{\{0,1\}^V} [(\nabla_{xy} f)(\eta)]^2 \, d\nu_\alpha(\eta).$

Weakly asymmetric exclusion process: Let $H : [0,T] \times V \to \mathbb{R}$ and $H_t = H(t, \cdot)$. Generator

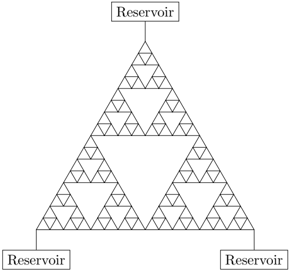
$$(\mathcal{L}^{\mathrm{EX}}_{(G,\mathbf{c}),H} f)(\eta) = \sum_{xy \in E} c_{xy} \psi_{xy}(H_t, \eta) (\nabla_{xy} f)(\eta). \quad f : \{0,1\}^V \to \mathbb{R},$$

where $\psi_{xy}(H, \eta) = \eta(x)[1 - \eta(y)] e^{H(y) - H(x)} + \eta(y)[1 - \eta(x)] e^{H(x) - H(y)}$.

Boundary-driven exclusion process

Declare a subset ∂V of V to be the boundary set.
Assume WLOG that $c_{aa'} = 0$ for each $a, a' \in \partial V$.
For each $a \in \partial V$, let $\lambda_+(a), \lambda_-(a) \in (0, \infty)$.

A **birth-and-death chain** added to each $a \in \partial V$.
At rate $\lambda_+(a)$, $\eta(a) = 0 \to \eta(a) = 1$ (birth).
At rate $\lambda_-(a)$, $\eta(a) = 1 \to \eta(a) = 0$ (death).



The boundary-driven exclusion process has generator

$$\mathcal{L}^{\mathrm{bEX}}_{(G,\mathbf{c})} = \mathcal{L}^{\mathrm{EX}}_{(G,\mathbf{c})} + \mathcal{L}^{\mathrm{b}}_{\partial V},$$

where

$$(\mathcal{L}^{\mathrm{b}}_{\partial V} f)(\eta) = \sum_{a \in \partial V} [\lambda_+(a)(1-\eta(a)) + \lambda_-(a)\eta(a)][f(\eta^a) - f(\eta)], \quad f : \{0,1\}^V \to \mathbb{R},$$

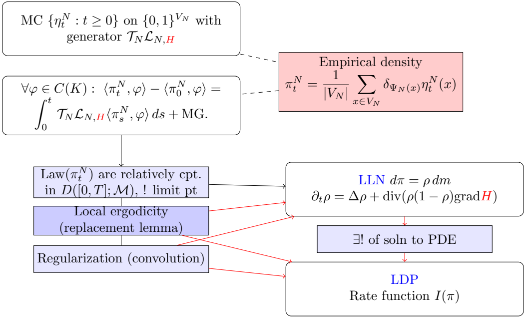
$$\eta^a(z) = \begin{cases} 1-\eta(a), & \text{if } z = a, \\ \eta(z), & \text{otherwise.} \end{cases}$$

Roadmap towards the hydrodynamic limit

[Guo, Kipnis, Landim, Olla, Papanicolaou, Varadhan, $\cdots$]

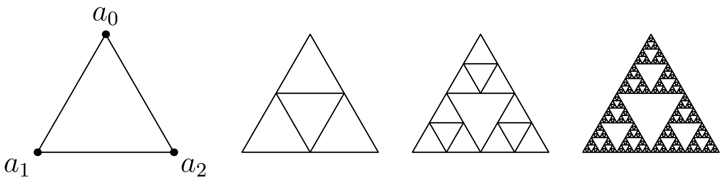
Random walk $(X_t^{(N)})$ on graph Γ_N. Isometrically embed Γ_N into a common compact space K via map $\Psi_N : \Gamma_N \to K$. Assume $\Psi_N(X_{\mathcal{T}_N t}^{(N)}) \Rightarrow Y_t$, diffusion process on K.

- $\mathcal{T}_N$: expected exit time of RW on Γ_N.
- $\mathcal{L}_{N,H}$: Generator of the (boundary-driven) exclusion process on Γ_N.
- H: Weak asymmetry in the rate, needed to obtain LDP.



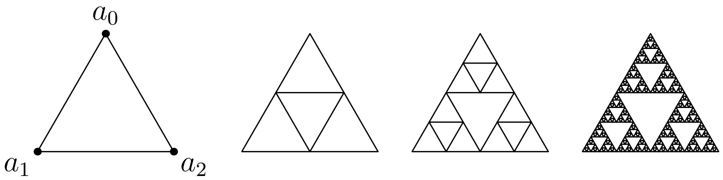
- This program has been carried out on $\mathbb{Z}^d$ since the 80's.
- **Challenge:** Extend the program to non-translationally-invariant, and possibly energy singular, spaces. Will use the Sierpinski gasket as a concrete example, but many arguments are expected to admit generalization.

Brownian motion on SG



- SG_N: Level-N Sierpinski gasket graph.
- m_N: self-similar measure on SG_N, assigns weight $\frac{1}{3^N}\frac{2}{3}$ to each vertex, except the three boundary points $\{a_0, a_1, a_2\}$ which receives weight $\frac{1}{3^N}\frac{1}{3}$.
- m_N converges weakly to m, the standard self-similar probability measure (with Hausdorff dimension $\log_2 3$), on the limit fractal K.
- $(X_t^N)_{t\geq 0}$: symmetric random walk process on SG_N.
- [Goldstein '87, Kusuoka '87, Barlow–Perkins '88]: Probability on fractals
  $X^N_{5^N t} \underset{N\to\infty}{\longrightarrow} B_t$, called a Brownian motion on SG.

Brownian motion on SG



- [Kigami '89+]: Analysis on fractals
  Write down the Dirichlet energy on SG_N, renormalized by $(5/3)^N$:

$$\mathcal{E}_N(f) = \left(\frac{5}{3}\right)^N \sum_{x\sim y} [f(x) - f(y)]^2 \qquad (f : K \to \mathbb{R})$$

  Then $\{\mathcal{E}_N\}_N$ is a monotone increasing sequence, and hence has a limit $\mathcal{E}$.
- Let $\mathcal{F}$ be the domain of $\mathcal{E}$. $(\mathcal{E}, \mathcal{F})$ is a strongly local regular Dirichlet form.
- **Operator convergence:** If Δ_N denotes the graph Laplacian on SG_N, then can prove the pointwise formula $\Delta = \frac{3}{2} \lim_{N\to\infty} 5^N \Delta_N$, where Δ is the generator of B_t.
- **Fact 1:** $\mathrm{dom}\Delta \subset \mathcal{F} \subset C(K)$. If $\mathcal{F}^*$ be the $L^2(K)$-dual of $\mathcal{F}$, then $\mathcal{M}(K) \subset \mathcal{F}^*$.
- **Fact 2:** "$\mathcal{E}(f) = \int_K |\nabla f|^2\, dm$" should be understood in terms of energy measure:
  $\mathcal{E}(f) = \int_K d\Gamma(f, f)$.
  SG is a prime example of a (energy) singular space. A prominent recent example of a singular space is diffusion on 2D Liouville quantum gravity [Garban–Rhodes–Vargas '14].

A more general framework: Resistance forms

Definition. [Kigami, early 2000's]
Let K be a nonempty set. A **resistance form** $(\mathcal{E}, \mathcal{F})$ on K is a pair such that

1. $\mathcal{F}$ is a vector space of $\mathbb{R}$-valued functions on K containing the constants, and $\mathcal{E}$ is a nonnegative definite symmetric quadratic form on $\mathcal{F}$ satisfying
$$\mathcal{E}(u,u) = 0 \;\Leftrightarrow\; u \text{ is constant.}$$
2. $\mathcal{F}/\{\text{constants}\}$ is a Hilbert space with norm $\mathcal{E}(u,u)^{1/2}$.
3. Given a finite subset $V \subset K$ and a function $v : V \to \mathbb{R}$, there is $u \in \mathcal{F}$ s.t. $u|_V = v$.
4. For $x, y \in K$,
$$R(x,y) := \sup\left\{ \frac{[u(x)-u(y)]^2}{\mathcal{E}(u,u)} : u \in \mathcal{F},\ \mathcal{E}(u,u) > 0 \right\} < \infty.$$
5. If $u \in \mathcal{F}$, then $\bar{u} := 0 \vee (u \wedge 1) \in \mathcal{F}$ and $\mathcal{E}(\bar{u},\bar{u}) \le \mathcal{E}(u,u)$.

(K, R) is a metric space. Can always assumed to be complete.

Note that Item 4 implies that $|u(x)-u(y)| \le R(x,y)^{1/2}\mathcal{E}(u,u)^{1/2}$, which then implies the Sobolev embedding $\mathcal{F} \subset C(K)$.

The classical Dirichlet form in $\mathbb{R}^n$ is a resistance form iff $n = 1$.

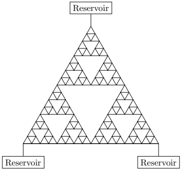
Resistance space = space K equipped with a resistance form $(\mathcal{E}, \mathcal{F})$

Standing Assumption. (K, R) is compact and connected; μ is a finite Borel measure on K.

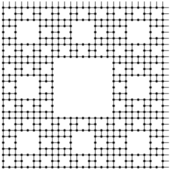
Then

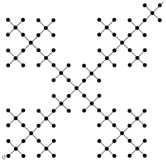
- Every function in $C(K)$ is bounded; thus $\mathcal{F}$ is an algebra under pointwise multiplication.
- $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(K, \mu)$. [Think: $\mathcal{E}(u, v) =$ "$\int_K \nabla u \cdot \nabla v \, d\mu$".]

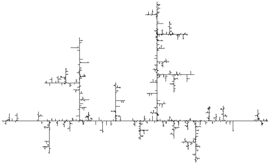
Examples (beyond the 1D interval)



Sierpinski gasket


Sierpinski carpet


Vicsek tree


Random dendrite [by David Croydon]

- If $B \subset K$ is a boundary, let $\mathcal{F}_0 = \{u \in \mathcal{F} : u|_B = 0\}$.
- **Assumption enforced.** The Dirichlet problem with boundary condition on B has a unique solution, in the sense of Dirichlet forms.

Problem: Closing the hydrodynamic equation

On SG_N, consider the **empirical density** measure

$$\pi_t^N := \frac{1}{3^N}\left(\frac{2}{3}\sum_{x\in V_N\setminus V_0}\eta_t^N(x)\mathbb{1}_x + \frac{1}{3}\sum_{a\in V_0}\eta_t^N(a)\mathbb{1}_a\right).$$

Let Q^N be the law of the Markov process generated by $\mathcal{L}^{\rm bEX}_{N,H}$, accelerated by 5^N and started from the initial distribution η_0^N.

Assume that $\langle \pi_0^N, F\rangle \to \int_K F\rho_0\,dm$ for some continuous density profile $\rho_0 : K \to (0,1)$.

By Dynkin's formula, under Q^N we have, for all test functions $F \in L^2(0,T,\mathcal{F}_0)$

$$\langle \pi_t^N, F\rangle = \langle \pi_0^N, F\rangle + \frac{2}{3}\int_0^t \langle \pi_s^N, \Delta_N F\rangle\,ds + \frac{2}{3}\int_0^t \sum_{a\in V_0}\eta_s^N(a)(\partial_N^\perp F)(a)\,ds$$
$$+\frac{2}{3}\int_0^t \sum_{xy\in E_N}\chi(\eta_s^N, xy)[H_s(x)-H_s(y)][F_s(x)-F_s(y)]\,ds + \underbrace{M_t^{N,F}}_{\text{martingale}}.$$

- $(\Delta_N F)(x) = \frac{3}{2}\frac{5^N}{3^N}\sum_{y\sim x}[F(y)-F(x)]$: (renormalized) Laplacian.
- $(\partial_N^\perp F)(a) = \frac{5^N}{3^N}\sum_{y\sim a}[F(y)-F(a)]$: (renormalized) normal derivative at the boundary point $a\in V_0$.
- $\chi(\eta, xy) = \eta(x)[1-\eta(y)] + \eta(y)[1-\eta(x)]$ is the conductivity of the exclusion process.

Problem: Closing the hydrodynamic equation (cont.)

$$\langle \pi_t^N, F\rangle = \langle \pi_0^N, F\rangle + \frac{2}{3}\int_0^t \langle \pi_s^N, \Delta_N F\rangle\, ds + \frac{2}{3}\int_0^t \sum_{a\in V_0} \eta_s^N(a)\, (\partial_N^\perp F)(a)\, ds$$

$$+ \frac{2}{3}\int_0^t \sum_{xy\in E_N} \chi(\eta_s^N, xy)\, [H_s(x) - H_s(y)][F_s(x) - F_s(y)]\, ds + \underbrace{M_t^{N,F}}_{\text{martingale}} .$$

Goal: Show that $\{Q^N\}$ is relatively compact, and the limit point Q^* concentrates on a.c. trajectories $\pi_t = \rho_t\, dm$ with $\rho \in L^2(0, T, \mathcal{F})$ and

$$\langle \pi_t, F\rangle = \langle \pi_0, F\rangle + \frac{2}{3}\int_0^t \langle \pi_s, \Delta F\rangle\, ds$$

$$+ \frac{2}{3}\int_0^t \sum_{a\in V_0} \bar\rho(a)\, (\partial^\perp F)(a)\, ds + \frac{2}{3}\int_0^t \langle \boldsymbol{\chi}(\rho_s)\, \partial H_s, \partial F\rangle_{\mathcal{H}}\, ds.$$

- $\boldsymbol{\chi}(\rho) := \rho(1-\rho)$: conductivity in the exclusion process.
- $\bar\rho(a) = \frac{\lambda_+(a)}{\lambda_+(a)+\lambda_-(a)}$: steady-state particle density at the boundary point $a \in V_0$.
- $\mathcal{H}$ is a Hilbert space of 1-forms on SG, induced by the Dirichlet form $(\mathcal{E}, \mathcal{F})$ for Brownian motion, following [Cipriani–Sauvageot '03, Hinz–Röckner–Teplyaev '13].
- $\partial : \mathcal{F} \to \mathcal{H}$ is the (abstract) gradient operator.

Problem: Closing the hydrodynamic equation (cont.)

$$\langle \pi_t^N, F\rangle = \langle \pi_0^N, F\rangle + \frac{2}{3}\int_0^t \langle \pi_s^N, \Delta_N F\rangle\, ds + \frac{2}{3}\int_0^t \sum_{a\in V_0} \eta_s^N(a)\, (\partial_N^\perp F)(a)\, ds$$

$$+ \frac{2}{3}\int_0^t \sum_{xy\in E_N} \chi(\eta_s^N, xy)\, [H_s(x)-H_s(y)][F_s(x)-F_s(y)]\, ds + \underbrace{M_t^{N,F}}_{\text{martingale}}.$$

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$$+ \frac{2}{3}\int_0^t \sum_{a\in V_0} \bar\rho(a)\, (\partial^\perp F)(a)\, ds + \frac{2}{3}\int_0^t \langle \boldsymbol{\chi}(\rho_s)\, \partial H_s, \partial F\rangle_{\mathcal{H}}\, ds.$$

Observe that this is the weak formulation of a nonlinear parabolic heat eqn

$$\begin{cases} \partial_t \rho_t = \frac{2}{3}\Delta\rho_t - \frac{2}{3}\partial^*\left(\boldsymbol{\chi}(\rho_t)\partial H_t\right) & \text{on } (0,T)\times K\setminus V_0, \\ \rho(0,\cdot) = \rho_0 & \text{on } K\setminus V_0, \\ \rho(t,\cdot)|_{V_0} = \bar\rho & \text{on } (0,T)\times V_0. \end{cases}$$

Problem: Closing the hydrodynamic equation (cont.)

$$\langle \pi_t^N, F\rangle = \langle \pi_0^N, F\rangle + \frac{2}{3}\int_0^t \langle \pi_s^N, \Delta_N F\rangle\, ds + \frac{2}{3}\int_0^t \sum_{a\in V_0} \eta_s^N(a)\, (\partial_N^\perp F)(a)\, ds$$
$$+ \frac{2}{3}\int_0^t \sum_{xy\in E_N} \chi(\eta_s^N, xy)\, [H_s(x)-H_s(y)][F_s(x)-F_s(y)]\, ds + \underbrace{M_t^{N,F}}_{\text{martingale}}.$$

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$$\langle \pi_t, F\rangle = \langle \pi_0, F\rangle + \frac{2}{3}\int_0^t \langle \pi_s, \Delta F\rangle\, ds$$
$$+ \frac{2}{3}\int_0^t \sum_{a\in V_0} \bar\rho(a)\, (\partial^\perp F)(a)\, ds + \frac{2}{3}\int_0^t \langle \boldsymbol{\chi}(\rho_s)\, \partial H_s, \partial F\rangle_{\mathcal{H}}\, ds.$$

Key issues: Terms on the RHS are NOT ALL in terms of the empirical density π^N. Need to make the following replacements:

- Conductivity term: Replace $\chi(\eta_s^N,\cdot)$ by $\boldsymbol{\chi}\left(\mathrm{Av}_{B(\cdot, r_\epsilon N)}[\eta_s^N]\right)$ and then by $\boldsymbol{\chi}(\rho_s)$.
- Boundary term: Replace $\eta_s^N(a)$ by $\bar\rho(a)$.

Local ergodicity: the statement

Basic idea: Replace functionals of η_t^N by coarse-grained functionals of the empirical density π_t^N, with negligible cost in the scaling limit.

Call $\phi : V(\Gamma) \times \{0,1\}^{V(\Gamma)} \to \mathbb{R}$ is a **local function bundle** if $\exists r \in (0,\infty)$ such that $\phi(x,\cdot)$ depends only on $\{\eta(z) : z \in B(x,r)\}$.

- Examples: $\phi(x,\eta) = \eta(x)$, $\phi(x,\eta) = \sum_{y\sim x} \eta(x)\eta(y)$.

Given ϕ and x, define the global average $\Phi_x(\alpha) = \int \phi(x,\eta)\, d\nu_\alpha(\eta) \quad (\alpha \in [0,1])$.
Let

$$U_{N,\epsilon}(x,\eta) := \phi(x,\eta) - \Phi_x\left(\mathrm{Av}_{B(x,r_\epsilon N)}[\eta]\right).$$

$\mathbb{P}_\alpha^N$: law of (η_t^N) with generator $\mathcal{T}_N \mathcal{L}^{\mathrm{EX}}_{(\Gamma_N,\mathbf{c})}$, started from the product Bernoulli measure ν_α.

Local ergodicity (*a.k.a.* local equilibrium, replacement lemma)

For each $T>0$ and each $\delta>0$,

$$\limsup_{\epsilon\downarrow 0}\limsup_{N\to\infty}\sup_{x\in V_N}\frac{1}{\mathcal{V}_N}\log\mathbb{P}_\alpha^N\left\{\left|\int_0^T U_{N,\epsilon}(x,\eta_t^N)\,dt\right|>\delta\right\} = -\infty.$$

Euclidean case: For $([0,N]\cap\mathbb{Z})^d$, $\mathcal{T}_N = N^2$, $\mathcal{V}_N = N^d$. **SG:** $\mathcal{T}_N \asymp 5^N$, $\mathcal{V}_N \asymp 3^N$.

For the proof of LDP we need this superexponential estimate.
Via a tilting argument one may change the measure from $\mathbb{P}_\alpha^N$ to Q^N and obtain the same local ergodic statement.

$(\Gamma, \mathbf{c})$ is an infinite, locally finite, connected weighted graph.
Fix $o \in V(\Gamma)$, and let $(r_N)_N$ be an increasing sequence of radii $\nearrow \infty$.
Let $(\Gamma_N, \mathbf{c})$ denote the weighted graph $B(o, r_N)$ endowed with conductance $\mathbf{c}$.
Let $(\mathcal{T}_N)_N$ and $(\mathcal{V}_N)_N$ be two increasing sequences in $\mathbb{R}$.
(In practice: $\mathcal{T}_N$ is the expected exit time from $B(o, r_N)$, and $\mathcal{V}_N$ is the volume of $B(o, r_N)$.)

Assumption 1. $\liminf_{N\to\infty} \frac{\mathcal{T}_N}{\mathcal{V}_N} = \infty$.

Assumption 2. For each $x \in V(\Gamma)$,

$$\liminf_{\epsilon\downarrow 0} \liminf_{N\to\infty} \frac{\mathcal{T}_N}{\mathcal{V}_N} \frac{1}{\mathrm{diam}_R(B(x, r_{\epsilon N}))} = \infty.$$

Here $\mathrm{diam}_R(A)$ is the diameter of A in the **effective resistance** metric $R_{\rm eff}(\cdot,\cdot)$ on $(\Gamma, \mathbf{c})$:

$$[R_{\rm eff}(x,y)]^{-1} = \inf\left\{ \sum_{zw\in E} c_{zw}[h(z)-h(w)]^2 \,\middle|\, h: V \to \mathbb{R},\ h(x)=1,\ h(y)=0 \right\}.$$

Theorem (C. '17). Under **Assumptions 1+2**, local ergodicity holds in the exclusion process.

For each $T>0$ and each $\delta>0$,

$$\limsup_{\epsilon\downarrow 0} \limsup_{N\to\infty} \sup_{x\in V(\Gamma_N)} \frac{1}{\mathcal{V}_N} \log \mathbb{P}^N_\alpha \left\{ \left| \int_0^T U_{N,\epsilon}(x, \eta^N_t)\, dt \right| > \delta \right\} = -\infty.$$

Condition [E]. $\limsup_{N\to\infty} \frac{|V(\Gamma_N)|}{\mathcal{V}_N} < \infty$.

Condition [BR]. There exist $\gamma, \gamma' \in [1,\infty)$ such that for all $a \in \partial V$,

$$\gamma^{-1} \le \frac{\lambda_+(a)}{\lambda_-(a)} \le \gamma \quad \text{and} \quad (\gamma')^{-1} \le \frac{\lambda_+(a)}{c_a} \le \gamma'.$$

Identify a boundary set ∂V_N for each Γ_N.

Let $\rho_N : V_N \to \mathbb{R}$ be the unique harmonic extension of $\rho_N(a) := \frac{\lambda_+(a)}{\lambda_+(a)+\lambda_-(a)}$, $a \in \partial V_N$, to V_N.

Assumption 3. The sequence of boundary rates $(\{\lambda_\pm(a) : a \in \partial V_N\})_N$ is chosen such that

$$\limsup_{N\to\infty} \frac{\mathcal{T}_N}{\mathcal{V}_N} \sum_{a\in\partial V(\Gamma_N)} \Big| \underbrace{\sum_{y\in V(\Gamma_N)} c_{ay}[\rho_N(y)-\rho_N(a)]}_{=i_{\rho_N}(a)\,=\,\text{electric flow into } a} \Big| < \infty \quad \text{and} \quad \limsup_{N\to\infty} \frac{\mathcal{T}_N}{\mathcal{V}_N} \mathcal{E}^{\rm el}_{(\Gamma_N,\mathbf{c})}(\rho_N) < \infty.$$

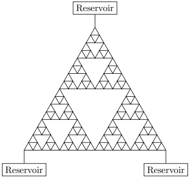
Remark. In terms of the trace of the RW process to ∂V_N, Assumption 3 says that the L^1- and L^2-energy norms of ρ_N, rescaled by $\mathcal{T}_N/\mathcal{V}_N$, are bounded as $N \to \infty$.

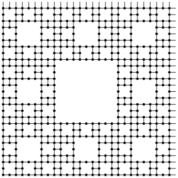
Theorem (C. '17). Under **[E]**, **[BR]**, and **Assumptions 1+2+3**, local ergodicity holds in the boundary-driven exclusion process.

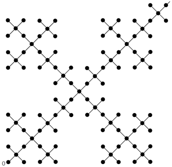
Remarks & applications

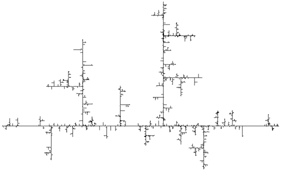
All assumptions derive from potential theory of random walks. Nothing is assumed about the spatial symmetries of the underlying space (weighted graph).

Assumptions 1+2 apply to any (very) **strongly recurrent** weighted graph in the sense of Delmotte, Barlow, and Telcs. (*SG*, 2D *SC*, Vicsek trees, continuum random trees, random graphs arising from critical percolation, ...) Local ergodicity holds on these spaces (new result).





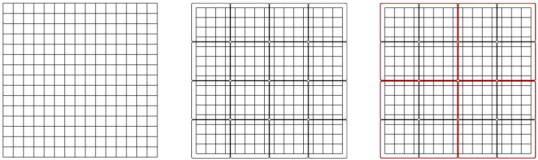




Limitation: Assumption 1 fails on transient or weakly recurrent weighted graphs, *i.e.*, spaces which are unbounded in the resistance metric. In a sense this is a low-dimensional result.

Remark. Our Assumptions 1+2 appear closely related to the ones used by Croydon '16 to obtain convergence of diffusion processes along a convergent sequence of metric measure spaces in the Gromov-Hausdroff-vague topology. (May be unified using Kigami's resistance forms.)

Technical estimates: 1-block and 2-blocks estimates



Strategy [*cf.* Kipnis–Olla–Varadhan '89]: implement a two-scale coarse-graining procedure.

$$U_{N,\epsilon}(x,\eta) := \underbrace{\left[\phi(x,\eta) - \Phi_x\left(\mathrm{Av}_{\Lambda_j(x)}[\eta]\right)\right]}_{U^{(1)}_{N,j} \quad \text{1-block}} + \underbrace{\left[\Phi_x\left(\mathrm{Av}_{\Lambda_j(x)}[\eta]\right) - \Phi_x\left(\mathrm{Av}_{B(x,r_{\epsilon N})}[\eta]\right)\right]}_{U^{(2)}_{N,j,\epsilon} \quad \text{2-blocks}}$$

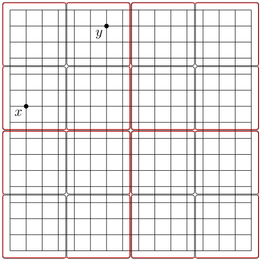
- j sets the microscopic scale.
- $\epsilon \in [0,1]$ sets the macroscopic aspect ratio.
- Ordering of limits: $N \to \infty$, then $\epsilon \downarrow 0$, then $j \to \infty$.

We separately show that $U^{(1)}_{N,j}$ and $U^{(2)}_{N,j,\epsilon}$ vanishes in the said limit with probability superexponentially close to 1. To do so we need to estimate the largest eigenvalue $\lambda^{(i)}_N$ of $\mathcal{T}_N \mathcal{L}^{\mathrm{EX}}_{(\Gamma_N,\mathbf{c})} \pm \mathcal{V}_N U^{(i)}$, $i \in \{1,2\}$, with respect to ν_α, using the Feynman-Kac formula.

For the 2BE to be effective, the energy cost of moving between points x and y in any two micro blocks inside a macro block,

$$\int_{\{0,1\}^V} [(\nabla_{xy} f)(\eta)]^2 \, d\nu_\alpha(\eta),$$

should scale diffusively.



Problem: Due to exclusion, when transferring a particle from x to y one also has to move over many "obstacles" along the path!

- $\mathbb{Z}^d$: Just pick a shortest path, and carry out a sequence of nearest-neighbor "spin swaps," and calculate the energy cost associated with this procedure. Then use Cauchy-Schwarz and the translation/rotation invariance of the exclusion process energy to obtain the diffusive scaling. [Kipnis–Olla–Varadhan '89]
- This "spin swaps along a chosen path" argument was also used by Diaconis–Saloff-Coste '93 to obtain eigenvalue bounds in the exclusion process on a finite graph.
- Infinite graphs without spatial symmetry: Challenging! New ideas are needed. Can we exploit the connection to the random walk process?

The crux of 2BE: Moving particle lemma

Let $(G = (V, E), \mathbf{c} = (c_{zw})_{zw \in E})$ be a finite connected weighted graph.
$c_{\rm eff}(x, y) = [R_{\rm eff}(x, y)]^{-1}$ effective conductance.

Theorem (C., *ECP* '17). The "moving particle lemma" for the exclusion process
For all $f : \{0,1\}^V \to \mathbb{R}$,

$$\underbrace{\sum_{zw \in E} c_{zw} \int_{\{0,1\}^V} [(\nabla_{zw} f)(\eta)]^2 \, d\nu_\alpha(\eta)}_{=2\mathcal{E}^{\rm EX}(f)} \geq c_{\rm eff}(x, y) \underbrace{\int_{\{0,1\}^V} [(\nabla_{xy} f)(\eta)]^2 \, d\nu_\alpha(\eta)}_{\text{Cost of swapping configs } x \leftrightarrow y}.$$

Remark. Nothing is known in general about when equality is attained.

Harkens to...

Theorem (Dirichlet/Thomson 1867). For all $f : V \to \mathbb{R}$,

$$\underbrace{\sum_{zw \in E} c_{zw}[f(z) - f(w)]^2}_{=\mathcal{E}(f)} \geq c_{\rm eff}(x, y)[f(x) - f(y)]^2.$$

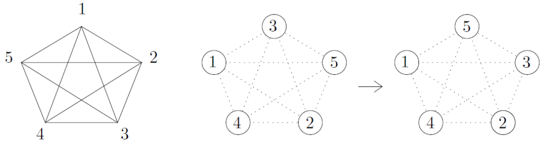
Equality is attained iff f is harmonic on $V \setminus \{x, y\}$.

Aldous' spectral gap problem, revisited

Where does the moving particle lemma come from?
Aldous' spectral gap conjecture (1992): $\lambda_{\mathrm{EX}}(G) = \lambda_{\mathrm{RW}}(G)$.

- An easy projection argument gives $\lambda_{\mathrm{EX}}(G) \leq \lambda_{\mathrm{RW}}(G)$.
- The nontrivial inequality to establish is $\lambda_{\mathrm{EX}}(G) \geq \lambda_{\mathrm{RW}}(G)$.

Aldous' suggestion: Consider the **interchange process** on G: n vertices, n labelled particles. Particles at x and y swap positions at rate c_{xy}.



$$(\mathcal{L}^{\mathrm{IP}} f)(\eta) = \sum_{xy \in E} c_{xy} [f(\eta^{xy}) - f(\eta)], \quad \mathcal{E}^{\mathrm{IP}}(f) = \sum_{xy \in E} c_{xy} \sum_{\eta \in \mathcal{X}_n} [f(\eta^{xy}) - f(\eta)]^2, \quad (f : \mathcal{X}_n \to \mathbb{R})$$

where $\mathcal{X}_n$ is the space of permutations on $\{1, 2, \cdots, n\}$, id'ed with V, and η^{xy} is obtained from η by transposing x and y.

Goal: Prove that $\lambda_{\mathrm{IP}}(G) \geq \lambda_{\mathrm{RW}}(G)$. (Since EX is a projection of IP, $\lambda_{\mathrm{EX}}(G) \geq \lambda_{\mathrm{IP}}(G)$.)

Insight: Network reduction

Remove vertices (and edges attached to them) without changing the effective conductance between any of the non-removed vertices.

- Remove the vertex $x \in V$ from $(G, \mathbf{c})$, as well as the edges attached to x. Call the reduced graph $G_x = (V_x, E_x)$.
  In the linear algebra language, we reduce the (probabilistic) Laplacian $\mathbf{L}$ to a new Laplacian $\mathbf{L}'$ (of one fewer dimension).
  This is attained by taking the Schur complement of the (x, x) block in $\mathbf{L}$:

$$\text{If } \mathbf{L} = \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{L}_{xx} \end{bmatrix}, \quad \text{then } \mathbf{L}' = \mathbf{X} - \mathbf{Y}(\mathbf{L}_{xx})^{-1}\mathbf{Z} = \mathbf{X} - \mathbf{Y}\mathbf{Z}. \qquad (\text{Recall } \mathbf{L}_{xx} = -1.)$$

- In component form, $\mathbf{L}'_{yz} = \mathbf{L}_{yz} - \mathbf{L}_{yx}\mathbf{L}_{xz}$ for $y, z \in V_x$. Since $\mathbf{L}^{(')}_{yz} = -p^{(')}_{yz} = -\frac{c^{(')}_{yz}}{c_y}$ whenever $y \neq z$, we see that the new conductances on E_x become

$$c'_{yz} = -c_y \mathbf{L}'_{yz} = -c_y(\mathbf{L}_{yz} - \mathbf{L}_{yx}\mathbf{L}_{xz}) = c_{yz} + \frac{c_{yx}c_{xz}}{c_x}.$$

Proposition. Upon network reduction (by removing x from $(G, \mathbf{c})$), the conductance on each edge in E_x increases by

$$\tilde{c}_{yz} := c'_{yz} - c_{yz} = \frac{c_{yx}c_{xz}}{c_x}.$$

This rule captures all familiar "physics textbook" circuit rules: Series law, Y-Δ transform, etc.

An algebraic miracle: the octopus inequality

Key idea: Upon network reduction by one vertex,
Energy lost from the removed edges $\geq$ Energy gained due to the increased conductances

Random walk process (electric networks)

$$\sum_{y \in V_x} c_{xy}[f(x) - f(y)]^2 \geq \sum_{yz \in E_x} \tilde{c}_{yz}[f(y) - f(z)]^2 \qquad (f : V \to \mathbb{R})$$

where equality is attained iff $(\mathbf{L}f)(x) = 0$.

Proof. High school algebra.

Interchange process: the **octopus inequality** of Caputo–Liggett–Richthammer *JAMS* '10

$$\sum_{y \in V_x} c_{xy} \sum_{\eta \in \mathcal{X}_n} [f(\eta^{xy}) - f(\eta)]^2 \geq \sum_{yz \in E_x} \tilde{c}_{yz} \sum_{\eta \in \mathcal{X}_n} [f(\eta^{yz}) - f(\eta)]^2 \qquad (f : \mathcal{X}_n \to \mathbb{R})$$

Proof. Clever use of (college-level?) linear algebra (especially Schur complements).

The octopus inequality implies $\lambda_{\rm IP}(G) \geq \lambda_{\rm RW}(G)$, and is key to the proof of Aldous' conjecture.

Take-away message: Energy is monotone decreasing along a sequence of network reductions.

Octopus inequality $\Rightarrow$ moving particle lemma [C. '17]

Fix a pair of vertices $x, y \in V$ in an n-vertex graph $(G, \mathbf{c})$.
Carry out network reductions one vertex at a time, until only x and y remain.

Random walk process (electric networks)

Applying the energy inequality to the sequence of network reductions, we find

$$\mathcal{E}^{\rm el}_{(G,\mathbf{c})}(f) \geq \mathcal{E}^{\rm el}_{(G_1,\mathbf{c}_1)}(f) \geq \cdots \geq \mathcal{E}^{\rm el}_{(G_{|V|-2},\mathbf{c}_{|V|-2})}(f) = \left(\mathbf{c}_{|V|-2}\right)_{xy} [f(x)-f(y)]^2.$$

Recognize that $\left(\mathbf{c}_{|V|-2}\right)_{xy} = c_{\rm eff}(x,y) = [R_{\rm eff}(x,y)]^{-1}$: the effective resistance is invariant under network reduction.

Thus

$$\mathcal{E}^{\rm el}_{(G,\mathbf{c})}(f) \geq c_{\rm eff}(x,y)[f(x)-f(y)]^2.$$

Fix a pair of vertices $x, y \in V$ in an n-vertex graph $(G, \mathbf{c})$.
Carry out network reductions one vertex at a time, until only x and y remain.

Interchange process

Applying the **octopus inequality** to the sequence of network reductions, we find

$$\mathcal{E}^{\rm IP}_{(G,\mathbf{c})}(f) \geq \mathcal{E}^{\rm IP}_{(G_1,\mathbf{c}_1)}(f) \geq \cdots \geq \mathcal{E}^{\rm IP}_{\left(G_{|V|-2},\mathbf{c}_{|V|-2}\right)}(f) = \left(\mathbf{c}_{|V|-2}\right)_{xy} \sum_{\eta\in\mathcal{X}_n} [f(\eta^{xy}) - f(\eta)]^2.$$

Recognize that $\left(\mathbf{c}_{|V|-2}\right)_{xy} = c_{\rm eff}(x,y) = [R_{\rm eff}(x,y)]^{-1}$: the effective resistance is invariant under network reduction.

Thus

$$\mathcal{E}^{\rm IP}_{(G,\mathbf{c})}(f) \geq c_{\rm eff}(x,y) \sum_{\eta\in\mathcal{X}_n} [f(\eta^{xy}) - f(\eta)]^2.$$

Fix a pair of vertices $x, y \in V$ in an n-vertex graph $(G, \mathbf{c})$.
Carry out network reductions one vertex at a time, until only x and y remain.

Exclusion process

Key: Particle number is conserved in the exclusion process, so $\{0,1\}^V = \bigoplus_{k=0}^n S_k^{\mathrm{EX}}$, where each chamber has k total particles.

Let $\pi_k : S^{\mathrm{IP}} \to S_k^{\mathrm{EX}}$ be the projection which outputs the configuration of the first k labelled particles in an IP configuration.

Lemma. If $\mathcal{E}^{\mathrm{EX}}_{(G,\mathbf{c}),\nu_\alpha}$ is the exclusion process Dirichlet energy w.r.t. ν_α, then

$$\mathcal{E}^{\mathrm{EX}}_{(G,\mathbf{c}),\nu_\alpha}(f) = \sum_{k=0}^n \binom{n}{k} \alpha^k (1-\alpha)^{n-k} \mathcal{E}^{\mathrm{IP}}_{(G,\mathbf{c})}(f_k \circ \pi_k),$$

where f_k is the orthogonal projection onto S_k^{EX}.

Finally, use the energy monotonicity of $\mathcal{E}^{\mathrm{IP}}$ under network reduction to conclude that

$$\mathcal{E}^{\mathrm{EX}}_{(G,\mathbf{c}),\nu_\alpha}(f) \geq c_{\mathrm{eff}}(x,y) \int_{\{0,1\}^V} [(\nabla_{xy} f)(\eta)]^2 \, d\nu_\alpha(\eta).$$

Towards the hydrodynamic PDE

Used local ergodicity to “close” the equation.

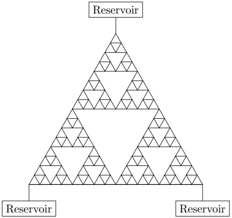
$$\langle \pi_t, F \rangle = \langle \pi_0, F \rangle + \frac{2}{3}\int_0^t \langle \pi_s, \Delta F \rangle \, ds$$
$$+ \frac{2}{3}\int_0^t \sum_{a \in V_0} \bar{\rho}(a) \, (\partial^\perp F)(a) \, ds + \frac{2}{3}\int_0^t \langle \boldsymbol{\chi}(\rho_s) \, \partial H_s, \partial F \rangle_{\mathcal{H}} \, ds.$$

- **One more input:** We need to establish $\exists$ and ! of the solution to the nonlinear parabolic PDE:
$$\begin{cases} \partial_t \rho_t = \frac{2}{3}\Delta \rho_t - \frac{2}{3}\partial^* \left(\boldsymbol{\chi}(\rho_t)\partial H_t\right) & \text{on } (0,T) \times K \setminus V_0, \\ \rho(0,\cdot) = \rho_0 & \text{on } K \setminus V_0, \\ \rho(t,\cdot)|_{V_0} = \bar{\rho} & \text{on } (0,T) \times V_0. \end{cases}$$

  Standard techniques (semigroup methods) are available in $\mathbb{R}^d$ and smooth manifolds [e.g. Evans' PDE]. However, on singular spaces these tools are NOT readily available, so we use the monotone operator method of J.-L. Lions.

  [In C.–Hinz–Teplyaev '18+ we address the solvability of such types of PDEs on resistance spaces.]

Hydrodynamic limit theorems on SG [C.–Hinz–Teplyaev '18+]



Law of large numbers for WASEP

Take any $H \in C([0,T],\mathcal{F}_0) \cap C^1((0,T),\mathcal{F}_0)$.

$$\begin{cases} \partial_t \rho_t = \frac{2}{3}\Delta\rho_t - \frac{2}{3}\partial^*\left(\chi(\rho_t)\partial H_t\right) & \text{on } (0,T)\times K \setminus V_0, \\ \rho(0,\cdot) = \rho_0 & \text{on } K \setminus V_0, \\ \rho(t,\cdot)|_{V_0} = \bar{\rho} & \text{on } (0,T)\times V_0. \end{cases}$$

Density large deviations principle

Let $\mathcal{FM}_+$ denote the collection of nonnegative Borel measures a.c. with respect to m, having density ρ which is bounded above by 1 and has finite energy, $\rho \in \mathcal{F}$.

Then for each closed set $\mathcal{C}$ and each open set $\mathcal{O}$ of the Skorokhod space $D([0,T],\mathcal{FM}_+)$,

$$\limsup_{N\to\infty} \frac{2}{3}\frac{1}{3^N}\log Q^N[\mathcal{C}] \leq -\inf_{\pi\in\mathcal{C}} I(\pi),$$

$$\liminf_{N\to\infty} \frac{2}{3}\frac{1}{3^N}\log Q^N[\mathcal{O}] \geq -\inf_{\pi\in\mathcal{O}} I(\pi).$$

The rate function I is the sum of a static contribution and a dynamic contribution I_0.

$$I_0(\pi) = \frac{1}{3}\int_0^T \int_K \chi\left(\frac{d\pi_t}{dm}\right) d\Gamma(H_t)\, dt \qquad (\pi \in \mathcal{FM}_+).$$

Here $d\Gamma(H_t)$ is the (Kusuoka) **energy measure** on K: it would equal $|\nabla H_t|^2\, dm$ on smooth spaces, but NOT on singular spaces.