Directed Polymers in Random (Heavy-Tail) Environment

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(joint works with Niccolò Torri)

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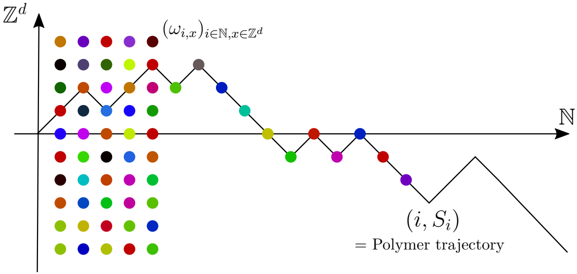
Montreal summer workshop :
challenges in probability and mathematical physics

The model

Take $(S_i)_{i \geqslant 0}$ a Simple Random Walk on $\mathbb{Z}^d$, law denoted $\mathbf{P}$.

Polymer trajectory = directed simple random walk $(i, S_i)_{i \geqslant 0}$.

Random environment = i.i.d. field $\omega = (\omega_{i,x})_{i \in \mathbb{N}, x \in \mathbb{Z}^d}$, law denoted $\mathbb{P}$.



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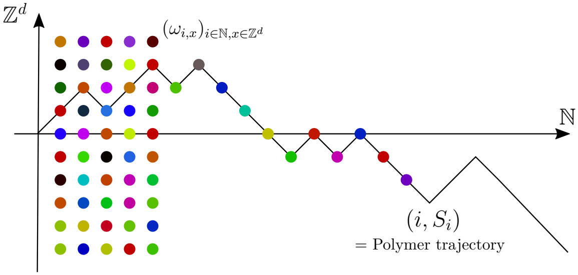

For $n \in \mathbb{N}$, ω fixed (quenched disorder), $\beta > 0$, define the Gibbs measure :

$$\frac{\mathrm{d}\mathbf{P}^{\omega}_{n,\beta}}{\mathrm{d}\mathbf{P}}(S) = \frac{1}{\mathbf{Z}^{\omega}_{n,\beta}} \exp\left(\beta \sum_{i=1}^{n} \omega_{i,S_i}\right)$$

A phase transition ? Localization of paths ?

Assume that $\lambda(\beta) := \log \mathbb{E}[e^{\beta\omega_{1,1}}] < +\infty$.

There are $\beta_c, \bar{\beta}_c$ such that

- For $\beta < \beta_c$, "weak disorder"
  Diffusive behavior of trajectories, Brownian scaling.
- For $\beta > \bar{\beta}_c$, "(very) strong disorder" :
  Localization in *favorite corridors*.

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$\mathbf{W}_{n,\beta}^{\omega} := \mathbf{Z}_{n,\beta}^{\omega} e^{-n\lambda(\beta)}$ is a positive martingale, and $\beta_c, \bar{\beta}_c$ are as follows :

$$\beta_c = \sup\{\beta, \lim_{n\to+\infty} \mathbf{W}_{n,\beta}^{\omega} > 0 \text{ a.s.}\},$$

$$\bar{\beta}_c = \sup\{\beta, \lim_{n\to+\infty} \frac{1}{n} \log \mathbf{W}_{n,\beta}^{\omega} = 0 \text{ a.s.}\}.$$

Open problem : $\beta_c = \bar{\beta}_c$ in dimension $d \geqslant 3$?

Questions in dimension $d = 1$

$\lambda(\beta) := \log \mathbb{E}[e^{\beta\omega_{1,1}}] < +\infty$.

Dimension $d = 1 \rightsquigarrow \bar{\beta}_c = 0$: localization for all $\beta > 0$.
Theorem : ($\omega_{i,x} \sim \mathcal{N}(0,1)$) there exists $x^{(n)} = x^{(n)}(\omega)$ such that

$$\liminf_{n\to+\infty} \mathbb{E}\mathbf{E}^{\omega}_{n,\beta}\Big[\frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{S_i = x_i^{(n)}\}}\Big] > 0\,.$$

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What can be said about path trajectories ?

superdiffusivity exponent : $\mathbb{E}\mathbf{E}^{\omega}_{n,\beta}|S_n| \approx n^{\xi}$. <u>Conjecture</u> : $\xi = 2/3$.

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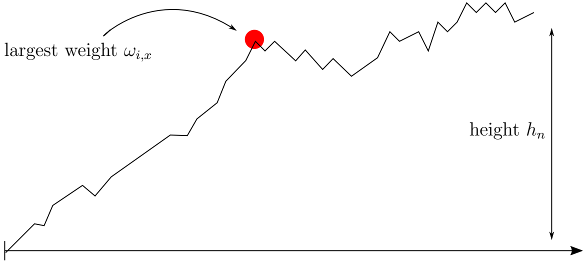
scaling limit ? Conjecture :

$$\frac{\log \mathbf{Z}^{\omega}_{n,\beta} - nf(\beta)}{c(\beta) n^{1/3}} \xrightarrow{(d)} \text{Tracy Widom}$$

The case of a heavy-tail environment

Assume that $\mathbb{P}(\omega_{1,1} > t) \sim t^{-\alpha}$, for some $\alpha > 0$.

Super-diffusivity exponent ?



Energetic gain : $\max \omega_{i,x} \approx (nh_n)^{1/\alpha}$

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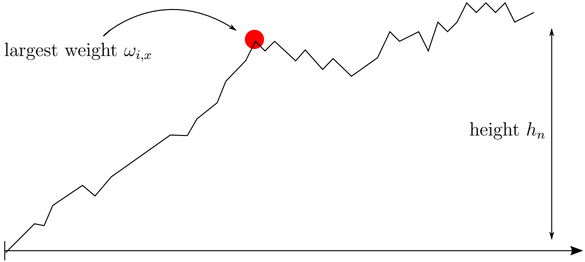

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Conjecture :

$$\xi = \frac{1+\alpha}{2\alpha-1}$$

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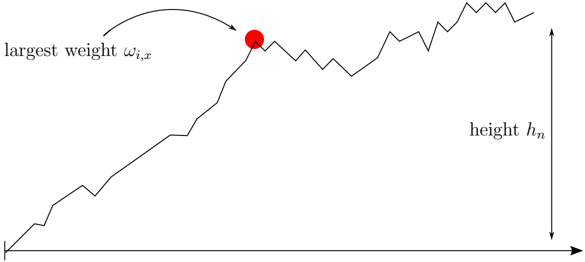

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Conjecture :

$\xi = \frac{1+\alpha}{2\alpha-1}$ if $\alpha \in (2,5)$; $\xi = \frac{2}{3}$ if $\alpha > 5$; $\xi = 1$ if $\alpha \in (0,2)$.

Finding when disorder "kicks-in"

Alberts, Khanin, Quastel '14 : take β go to 0 as $n \to +\infty$.

In the case $\lambda(\beta) = \log \mathbb{E}[e^{\beta \omega_{1,1}}] < +\infty$:

Take $\beta_n = \hat{\beta} n^{-1/4}$, then

$$\log \mathbf{Z}^{\omega}_{n,\beta_n} - n\lambda(\beta_n) \xrightarrow{(d)} \log \mathcal{Z}_{\sqrt{2}\hat{\beta}} .$$

Here, $\mathcal{Z}_{\sqrt{2}\hat{\beta}}$ is the solution of the Stochastic Heat Equation
(Interpolates between Gaussian as $\hat{\beta} \downarrow 0$ and Tracy-Widom as $\hat{\beta} \uparrow +\infty$.)

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Exactly when disorder "kicks-in" :
one still has $\mathbb{E}\mathbf{E}_{n,\beta_n}[|S_n|] = O(\sqrt{n})$, i.e. $\xi = 1/2$.

Finding when disorder "kicks-in", heavy-tail case

Case of a Heavy-tail disorder : $\mathbb{P}(\omega_{1,1} > t) \sim t^{-\alpha}$.

Dey, Zygouras '16 : take $\beta_n = \hat{\beta} n^{-\gamma}$, with :

$$\gamma = \tfrac{1}{4} \ \text{ if } \alpha > 6, \qquad \gamma = \tfrac{3}{2\alpha} \ \text{ if } \alpha \in (\tfrac{1}{2}, 6).$$

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Then

$$\log \mathbf{Z}_{n,\beta_n}^{\omega} - n\bar{\lambda}(\beta_n) \xrightarrow{(d)} \begin{cases} \log \mathcal{Z}_{\sqrt{2}\hat{\beta}} & \text{if } \alpha > 6, \\ \mathcal{N}(0, 2\pi\hat{\beta}^2) & \text{if } \alpha \in (2,6), \\ \mathcal{W}_{\beta}^{(\alpha)} & \text{if } \alpha \in (\tfrac{1}{2}, 2). \end{cases}$$

Again, disorder just "kicks-in" : $\xi = 1/2$

A general picture ?

Heavy-tail environment : $\mathbf{P}(\omega_{1,1} > t) \sim t^{-\alpha}$.
Weak-coupling : $\beta_n = \hat{\beta} n^{-\gamma}$.

Determine $\mathbb{E}\mathbf{E}_{n,\beta_n}|S_n| \approx n^{\xi}$?

On blackboard

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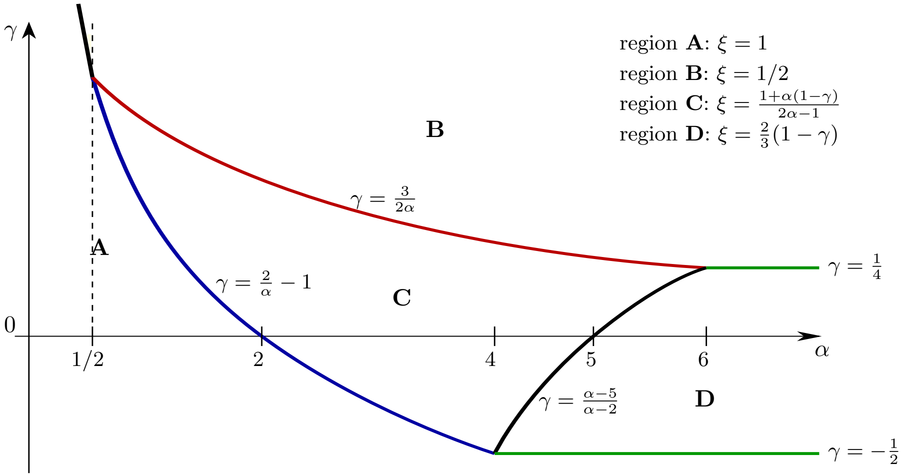

Results : B. & Torri, '18, case $\alpha \in (\frac{1}{2}, 2)$

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Theorem

If $\alpha \in (\frac{1}{2}, 2)$ *and* $\frac{2}{\alpha} - 1 < \gamma < \frac{3}{2\alpha}$, *then* $\xi = \frac{1+(1-\gamma)\alpha}{2\alpha-1}$.

$$\mathbb{P}\Big(\mathbf{P}^{\omega}_{n,\beta_n}\Big(\max_{1 \leq i \leqslant n} |S_i| \in [\varepsilon n^{\xi}, \varepsilon^{-1} n^{\xi}]\Big) \geqslant 1 - \varepsilon \Big) \geqslant 1 - \varepsilon$$

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We also identify the scaling limit :

$$\frac{1}{n^{2\xi-1}}\Big(\log \mathbf{Z}^{\omega}_{n,\beta_n} - c(\beta_n)\Big) \xrightarrow{(d)} \mathcal{T}_{\hat{\beta}} = \sup_{s:[0,1]\to\mathbb{R}} \Big\{ \hat{\beta} Energy(s) - Entropy(s) \Big\} .$$

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Theorem

If $\alpha \in (0, \frac{1}{2})$ and $\gamma = \frac{2}{\alpha} - 1$, then (Auffinger, Louidor '10, Torri '14) :

$$\frac{1}{n} \log \mathbf{Z}^{\omega}_{n,\beta_n} \xrightarrow{a.s.} \widetilde{\mathcal{T}}_{\hat{\beta}} \begin{cases} > 0 \quad (\textit{and } \xi = 1) & \textit{if } \hat{\beta} > \hat{\beta}_c(\omega), \\ = 0 & \textit{if } \hat{\beta} < \hat{\beta}_c(\omega). \end{cases}$$

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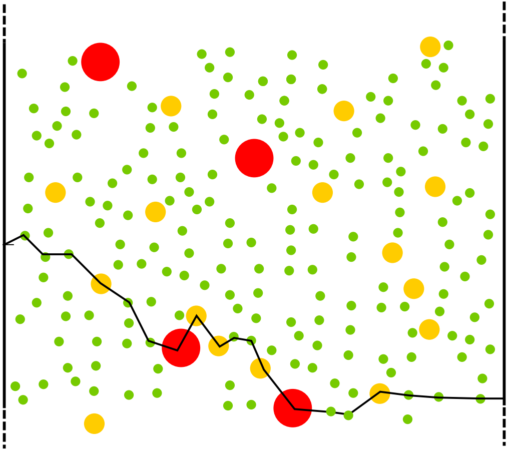
We prove that $\xi = 1/2$ if $\hat{\beta} < \hat{\beta}_c$.

Definition of the variational problem

Scaling limit of RW paths $= s : [0,1] \to \mathbb{R}$.

Continuum field = scaling limit of heavy-tail field
$\rightsquigarrow$ PPP on $\mathbb{R}_+ \times [0,1] \times \mathbb{R}$, of intensity $\alpha w^{-(1+\alpha)} \, \mathrm{d}w \, \mathrm{d}t \, \mathrm{d}x$.



Definition of the variational problem

Disordered field $= \mathcal{P}$ on $\mathbb{R}_+ \times [0,1] \times \mathbb{R}$, intensity $\alpha w^{-(1+\alpha)}\,\mathrm{d}w\,\mathrm{d}t\,\mathrm{d}x$
Continuum paths $= s : [0,1] \to \mathbb{R}$

Weight of a path :

$$\pi(s) = \pi^{(\mathcal{P})}(s) = \sum_{(w,t,x)\in\mathcal{P}, s(t)=x} w$$

Entropy of a path : ($\log \mathbf{P}(S_{tn} = x h_n) \sim \frac{x^2}{2t} h_n^2/n$ for $\sqrt{n \log n} \ll h_n \ll n$)

$$\mathrm{Ent}(s) = \frac{1}{2}\int_0^1 (s'(t))^2\,\mathrm{d}t.$$

Variational problem :

$$\mathcal{T}_{\hat\beta} = \mathcal{T}_{\hat\beta}(\mathcal{P}) = \sup_{s:[0,1]\to\mathbb{R}, \mathrm{Ent}(s)<+\infty} \Big\{ \hat\beta \pi(s) - \mathrm{Ent}(s) \Big\}.$$

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Theorem : $\mathcal{T}_{\hat\beta} \in (0,+\infty)$ for any $\hat\beta > 0$ if $\alpha \in (\frac{1}{2}, 2)$.

Difficulties

- Need to control that you cannot :
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  (2) accumulate a lot of small weights that compensate the entropic loss (need $\alpha < 2$).
- When $\xi = 1$, paths stay in a bounded box + paths are 1-Lipschitz !
  $\rightsquigarrow$ use standard Last-Passage Percolation (see Hambly, Martin '07)

We developed a new version of Last-Passage Percolation : paths are constrained to have a fixed entropy, rather than being 1-Lipschitz.