Anderson localization for high dimensional quasi-periodic operators with long-range interactions

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This is joint work with Svetlana Jitomirskaya and Yunfeng Shi. We study the long-range quasi-periodic operators on $\ell^2(\mathbb{Z}^d)$:

$$H = S(n, n') + \lambda v(x + n \otimes \omega)\delta_{nn'},$$

where S satisfies $|S(n, n')| \leq e^{-\rho|n-n'|}$, and

$$n \otimes \omega := (n_1\omega_{11}, \cdots, n_1\omega_{b_1 1}, \cdots, n_d\omega_{1d}, \cdots, n_d\omega_{b_d d}),$$

and v is a real analytic function on the $b = \sum_{i=1}^{d} b_i$ dimensional torus.

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- d is the dimension of operators ($\mathbb{Z}^d$). b is the dimension of torus (v is defined on $\mathbb{T}^b$).

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- Higher dimensional cases ($d = b \geq 2$): the potential is given by $V_{n_1,\cdots,n_b} = v(x_1 + n_1\omega_1, x_2 + n_2\omega_2, \cdots, x_b + n_b\omega_b)$. Bourgain-Goldstein-Schlag ($b = d = 2$) and Bourgain ($b = d \geq 3$).

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- The general b, d and long-range case is particularly important because it appears as Aubry-dual of general quasiperiodic Schrödinger operators.

In order to simplify the talk, we assume $d = 2$ and $b = 2 + 1 = 3$. It says that

$$V_{n_1,n_2} = v(x_1 + n_1\omega_1, x_2 + n_1\omega_2, x_3 + n_2\omega_3).$$

- Denote by $x^1 = (x_1, x_2)$ and $x^2 = x_3$. Assume the functions $v(x^1, \cdot)$, $v(\cdot, x^2)$ are nonconstant for any x^1, x^2.
- We call u a generalized eigenfunction (E corresponding generalized eigenvalue) of H if $Hu = Eu$ and

$$|u(n)| \leq C(1 + |n|^C).$$

Theorem 1 (Jitomirskaya-L-Shi)

For any $\delta > 0$, there is some $\lambda_0 = \lambda_0(\delta, v, b, d, \rho) > 0$ such that, for $|\lambda| \geq \lambda_0$ and $x \in \mathbb{T}^b$, there exists some $\Omega \subset \mathbb{T}^b$ with $\mathrm{Leb}(\mathbb{T}^b \setminus \Omega) \leq \delta$ such that, for $\omega \in \Omega$, all the generalized eigenfunctions of H decay exponentially.

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Corollary 2 (Jitomirskaya-L-Shi)

Suppose S is a Töplitz operator. Then under the assumption of above Theorem, H exhibits Anderson localization.

Remark:

- The assumption of v in 1D and multi-frequencies case (Bourgain and Bourgain-Goldstein) is v is not constant.
- The assumption of v in higher dimension cases-$d \geq 2$ (Bourgain-Goldstein-Schlag and Bourgain) is v is not constant in each variable.
- Two dimensions: Bourgain-Kachkovskiy and Bourgain-Jitomirskaya-Kachkovskiy.
- Applications in PDEs: Wang

General strategy

Two key steps:

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Two key steps:

- Large deviation theorem. It was done by delicate estimates on the Green's function for fixed energy, which relies on subharmonicity and the arithmetic conditions on the frequencies.
- Elimination of the energy and proof of Anderson localization. It was done by the semi-algebraic set theory (steep lemma). As a result, it is necessary to remove some extra frequencies.

Large deviation theorem. Under some arithmetic assumption of frequencies, the Green's functions are good except an exponentially small set of x (say e^{-N^γ}).

- Green's functions: $G_N = (R_\Lambda(H-E)R_\Lambda)^{-1}$, where Λ is an elementary regime ($\Lambda = [-N, N]^2$)
- "Good":

$$||G_N|| \leq e^{N^\kappa},$$
$$|G_N(n, n')| \leq e^{-\frac{\rho}{2}|n-n'|} \text{ for } |n-n'| \geq \frac{N}{10}.$$

Two remarks:

- The exceptional sets of x depend on energy E and frequencies ω.
- The goal is to remove frequencies such that the Anderson localization holds for fixed x.

Eliminating the energy and proof of Anderson localization

Pick a generalized eigenvalue E and corresponding eigenfunction ϕ.

- For some $\sigma > 0$,

$$\mathrm{dist}(E, \sigma(H_{[-N,N]^2})) \leq e^{-\sigma N}.$$

- Localization: Resolvent identity arguments+multi-scale analysis; Removing frequencies by steep Lemma.

Geometric structure of the exceptional set

- Suppose $G_N(\omega, x + n\omega)$ is good for all $N_1^{c_2} \leq |n| \leq N_1$, where $N_1 = e^{N^{c_1}}$. Then generalized eigenfunction is localized in regime $2N_1^{c_2} \leq |n| \leq \frac{1}{2}N_1$. (resolvent identity,multi-scale analysis)
- We expect to show that by removing a small set of ω, $G_N(\omega, x + n\omega)$ is good for all $N_1^{c_2} \leq |n| \leq N_1$.
- It relies on the geometric structure of the exceptional set ($e^{-N^{\gamma}}$).

Removing frequencies by steep lemma

We use $d = b = 1$ as an example.

Lemma 3

Let $A = \{(\omega, x) \in \mathbb{T}^2 : G_N(\omega, x)$ is bad$\} \leq e^{-N^\gamma}$. Suppose A has "nice" geometric structure. Then after removing a small set of frequencies ($N_1^{-c_3}$), the Green's function $G_N(\omega, x + n\omega)$ is good for all $N_1^{c_2} \leq |n| \leq N_1$.

Semi-algebraic set

Let $\{P_1, \cdots, P_s\} \subset \mathbb{P}[x_1, \cdots, x_n]$ be a family of real polynomials whose degrees are bounded by q. A (closed) semi-algebraic set $\mathcal{S}$ is given by an expression

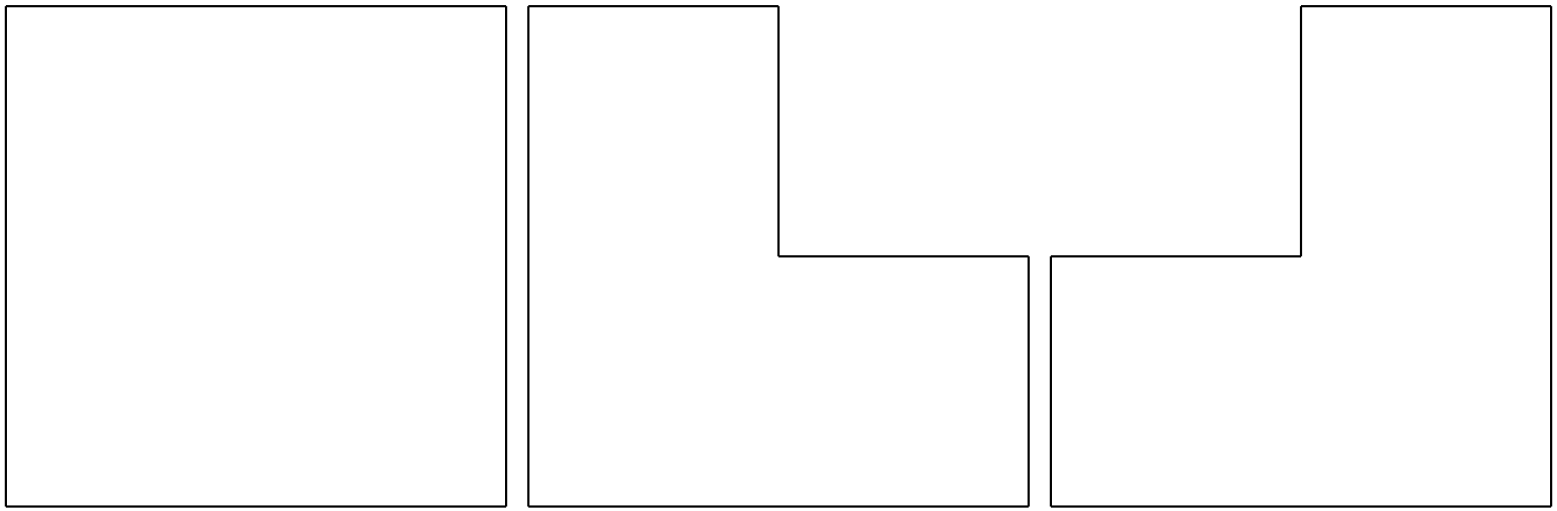
$$\mathcal{S} = \bigcup_j \bigcap_{\ell \in \mathcal{L}_j} \{x \in \mathbb{R}^n : \ P_\ell(x) s_{j\ell} 0\}, \tag{1}$$

where $\mathcal{L}_j \subset \{1, \cdots, s\}$ and $s_{j\ell} \in \{\geq, \leq, =\}$. The degree of $\mathcal{S}$, denoted by $\deg(\mathcal{S})$, is the smallest sq over all representations as in (1).

In higher dimensions, everything becomes complicate.

- For $d = b \geq 3$, Bourgain uses semi-algebraic set to obtain large deviation theorem even in the first step.
- For $d \geq 3$, there are a lot of resonances.
- We can not use steep lemma directly to remove frequencies.

Elementary regimes



Large Deviation Theorem

Theorem 4 (Jitomirskaya-L-Shi)

There exist constants $\gamma, \kappa \in (0,1)$ and large N_0 such that if $\log\log\lambda \geq N_0$ and $N \geq N_0$, there is some semi-algebraic set $\Omega_N \subset \mathbb{T}^3$ with $\deg(\Omega_N) \leq N^C$ such that the following holds: $\mathrm{Leb}(\mathbb{T}^3 \backslash \Omega_N) \leq \delta(\lambda)$ ($\delta(\lambda) \to 0$ as $\lambda \to \infty$). For any $\omega \in \Omega_N, E \in \mathbb{R}$, there exists some set $X_N = X_N(\omega, E) \subset \mathbb{T}^3$ such that

$$\mathrm{Leb}\{\theta \in \mathbb{T} : (x^1, \theta) \in X_N\} \leq e^{-N^\gamma},$$

$$\mathrm{Leb}\{\theta \in \mathbb{T}^2 : (\theta, x^2) \in X_N\} \leq e^{-N^\gamma}$$

and for $x \notin X_N$, the Green's function G_N is good.

- For $d = 1$ and $d = b = 2$, Ω_N does not depend on N. It has full Lebesgue measure, which is given by arithmetic conditions. For example, ω satisfies Diophantine condition for $d = 1$.
- Our proof is inspired by Bourgain ($d = b \geq 3$). We need to use semi-algebraic set (steep lemma) to prove LDT.
- The energy elimination is easier than LDT.

Proof of LDT

- Starting scale N_0.
- Increase the scale from N_0 to $N_1 = e^{N_0^{c_1}}$ with a hole for all x and E: Resolvent identity arguments+multi-scale analysis; Removing frequencies by steep Lemma.

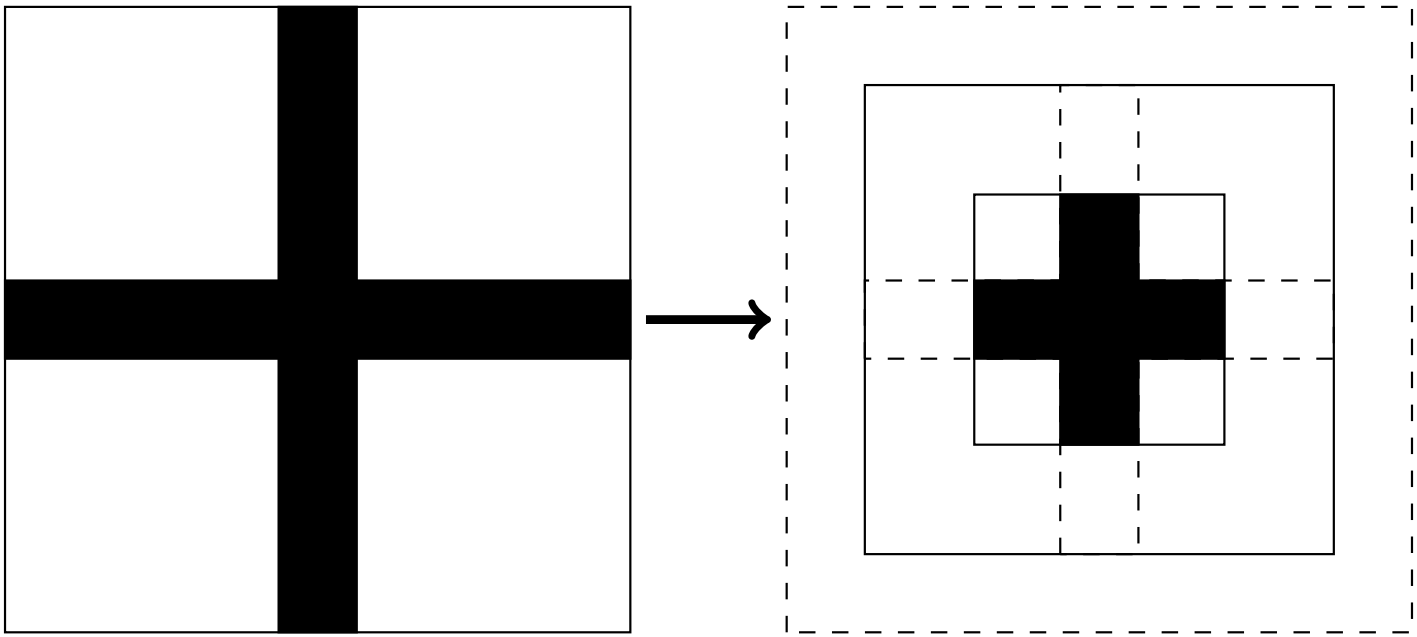
"Hole"

For all $(E, x) \in \mathbb{R} \times \mathbb{T}^3$, there is N satisfying ${N_1}^{c_4} < N < N_1^{c_5}$ such that if we let

$$U = [-N, N]^2 \setminus [-N^{\frac{1}{C}}, N^{\frac{1}{C}}]^2,$$

then

$$||G_U(E; x)|| \leq e^{N^{\kappa}},$$
$$||G_U(E; x)(n, n')|| \leq e^{-\frac{\rho}{2}|n-n'|} \text{ for } |n - n'| > \frac{N}{10}.$$



Matrix valued Cartan theorem

Except a small set of x ($e^{-N_1^{\gamma}}$), for any $n_0 \in [-N_1, N_1]^2$, there is N (depends on n_0) satisfying ${N_1}^{c_4} < N < N_1^{c_5}$ such that

$$||G_{\mathcal{E}_N}(E;x)|| \leq e^{N^{\kappa}},$$

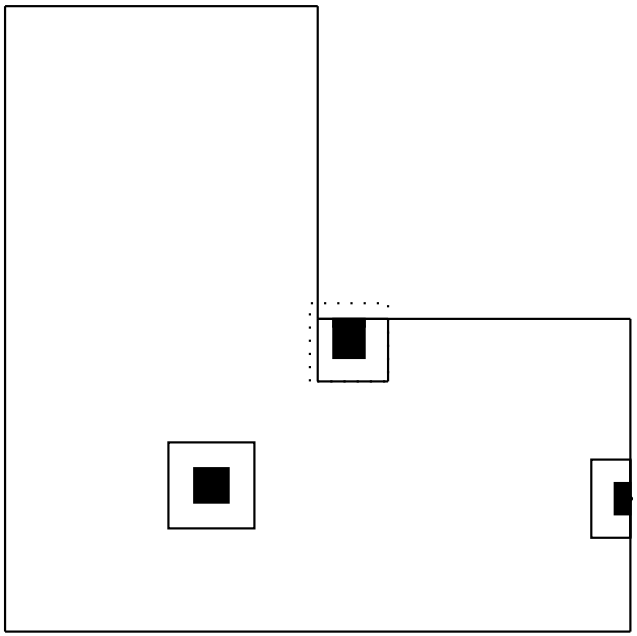
$$||G_{\mathcal{E}_N}(E;x)(n,n')|| \leq e^{-\frac{\rho}{2}|n-n'|} \text{ for } |n-n'| > \frac{N}{10},$$

where $\mathcal{E}_N$ has center n_0.

The Green's function at scale N_1 is good based on scale N

Difficulty: the size N changes as the center n_0 changes. We need extra work to deal with boundary cases.

- Modify the definition of semi-algebraic set ("good" of Green's function).
- Modify the "holes".



Thank you