Anderson localization for high dimensional quasi-periodic operators with long-range interactions

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Spectral Theory of Quasi-Periodic and Random Operators,
November 15, 2018
This is joint work with Svetlana Jitomirskaya and Yunfeng Shi. We study the long-range quasi-periodic operators on $\ell^2(\mathbb{Z}^d)$:

$$H = S(n, n') + \lambda \nu(x + n \otimes \omega) \delta_{nn'},$$

where $S$ satisfies $|S(n, n')| \leq e^{-\rho |n-n'|}$, and

$$n \otimes \omega := (n_1\omega_{11}, \cdots, n_1\omega_{b_11}, \cdots, n_d\omega_{1d}, \cdots, n_d\omega_{b_dd}),$$

and $\nu$ is a real analytic function on the $b = \sum_{i=1}^{d} b_i$ dimensional torus.
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- Töplitz operators: $S(n, n') = \hat{\phi}(n - n')$ and $\phi$ is a real analytic function on $\mathbb{T}^d$.
- $d$ is the dimension of operators ($\mathbb{Z}^d$). $b$ is the dimension of torus ($\nu$ is defined on $\mathbb{T}^b$).
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Anderson localization

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- One dimensional Schrödinger operators \((d = 1)\): the potential is given by \(V_n = v(x_1 + n\omega_1, x_2 + n\omega_2, \cdots, x_b + n\omega_b)\), where \(n \in \mathbb{Z}\). Bourgain-Goldstein \((d = 1, b = 2)\), Bourgain \((d = 1, b \geq 3)\).
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- Bourgain-Goldstein \((d = 1, b = 2)\), Bourgain \((d = 1, b \geq 3)\).
- Higher dimensional cases \((d = b \geq 2)\): the potential is given by \(V_{n_1, \ldots, n_b} = v(x_1 + n_1\omega_1, x_2 + n_2\omega_2, \cdots, x_b + n_b\omega_b)\). Bourgain-Goldstein-Schlag \((b = d = 2)\) and Bourgain \((b = d \geq 3)\).
Anderson localization

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- The general \(b, d\) and long-range case is particularly important because it appears as Aubry-dual of general quasiperiodic Schrödinger operators.
In order to simplify the talk, we assume \( d = 2 \) and \( b = 2 + 1 = 3 \). It says that

\[
V_{n_1, n_2} = v(x_1 + n_1\omega_1, x_2 + n_1\omega_2, x_3 + n_2\omega_3).
\]

- Denote by \( x^1 = (x_1, x_2) \) and \( x^2 = x_3 \). Assume the functions \( v(x^1, \cdot), v(\cdot, x^2) \) are nonconstant for any \( x^1, x^2 \).
- We call \( u \) a generalized eigenfunction (\( E \) corresponding generalized eigenvalue) of \( H \) if \( Hu = Eu \) and

\[
|u(n)| \leq C(1 + |n|^C).
\]
Theorem 1 (Jitomirskaya-L-Shi)

For any $\delta > 0$, there is some $\lambda_0 = \lambda_0(\delta, v, b, d, \rho) > 0$ such that, for $|\lambda| \geq \lambda_0$ and $x \in T^b$, there exists some $\Omega \subset T^b$ with $\text{Leb}(T^b \setminus \Omega) \leq \delta$ such that, for $\omega \in \Omega$, all the generalized eigenfunctions of $H$ decay exponentially.
Theorem 1 (Jitomirskaya-L-Shi)

For any $\delta > 0$, there is some $\lambda_0 = \lambda_0(\delta, v, b, d, \rho) > 0$ such that, for $|\lambda| \geq \lambda_0$ and $x \in \mathbb{T}^b$, there exists some $\Omega \subset \mathbb{T}^b$ with \( \text{Leb}(\mathbb{T}^b \setminus \Omega) \leq \delta \) such that, for $\omega \in \Omega$, all the generalized eigenfunctions of $H$ decay exponentially.

Corollary 2 (Jitomirskaya-L-Shi)

Suppose $S$ is a Töplitz operator. Then under the assumption of above Theorem, $H$ exhibits Anderson localization.
Remark:

- The assumption of $\nu$ in 1D and multi-frequencies case (Bourgain and Bourgain-Goldstein) is $\nu$ is not constant.
- The assumption of $\nu$ in higher dimension cases-$d \geq 2$ (Bourgain-Goldstein-Schlag and Bourgain) is $\nu$ is not constant in each variable.
- Two dimensions: Bourgain-Kachkovskiy and Bourgain-Jitomirskaya-Kachkovskiy.
- Applications in PDEs: Wang
General strategy

Two key steps:

- Large deviation theorem. It was done by delicate estimates on the Green’s function for fixed energy, which relies on subharmonicity and the arithmetic conditions on the frequencies.

- Elimination of the energy and proof of Anderson localization. It was done by the semi-algebraic set theory (steep lemma).

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Large deviation theorem. Under some arithmetic assumption of frequencies, the Green’s functions are good except an exponentially small set of $x$ (say $e^{-N^\gamma}$).

- Green’s functions: $G_N = (R_\Lambda (H - E) R_\Lambda)^{-1}$, where $\Lambda$ is an elementary regime ($\Lambda = [-N, N]^2$)

- “Good”:

\[
\| G_N \| \leq e^{N^\kappa},
\]

\[
| G_N(n, n') | \leq e^{-\frac{\rho}{2}|n-n'|} \text{ for } |n - n'| \geq \frac{N}{10}.
\]
Two remarks:

- The exceptional sets of $x$ depend on energy $E$ and frequencies $\omega$.
- The goal is to remove frequencies such that the Anderson localization holds for fixed $x$. 
Pick a generalized eigenvalue $E$ and corresponding eigenfunction $\phi$.

- For some $\sigma > 0$,
  \[ \text{dist}(E, \sigma(H_{[-N,N]^2})) \leq e^{-\sigma N}. \]

- Localization: Resolvent identity arguments + multi-scale analysis; Removing frequencies by steep Lemma.
Suppose $G_N(\omega, x + n\omega)$ is good for all $N_1^{c_2} \leq |n| \leq N_1$, where $N_1 = e^{N_1}$. Then generalized eigenfunction is localized in regime $2N_1^{c_2} \leq |n| \leq \frac{1}{2} N_1$. (resolvent identity, multi-scale analysis)

We expect to show that by removing a small set of $\omega$, $G_N(\omega, x + n\omega)$ is good for all $N_1^{c_2} \leq |n| \leq N_1$.

It relies on the geometric structure of the exceptional set $(e^{-N\gamma})$. 
We use $d = b = 1$ as an example.

Lemma 3

Let $A = \{(\omega, x) \in \mathbb{T}^2 : G_N(\omega, x) \text{ is bad}\} \leq e^{-N\gamma}$. Suppose $A$ has “nice” geometric structure. Then after removing a small set of frequencies $(N_1^{c_3})$, the Green’s function $G_N(\omega, x + n\omega)$ is good for all $N_1^{c_2} \leq |n| \leq N_1$. 

Wencai Liu Anderson Localization
Semi-algebraic set
Let \( \{P_1, \cdots, P_s\} \subset \mathbb{P}[x_1, \cdots, x_n] \) be a family of real polynomials whose degrees are bounded by \( q \). A (closed) semi-algebraic set \( S \) is given by an expression

\[
S = \bigcup_{j} \bigcap_{\ell \in \mathcal{L}_j} \{x \in \mathbb{R}^n : P_\ell(x) s_{j\ell} 0\},
\]

(1)

where \( \mathcal{L}_j \subset \{1, \cdots, s\} \) and \( s_{j\ell} \in \{\geq, \leq, =\} \). The degree of \( S \), denoted by \( \text{deg}(S) \), is the smallest \( sq \) over all representations as in (1).
In higher dimensions, everything becomes complicate.

- For $d = b \geq 3$, Bourgain uses semi-algebraic set to obtain large deviation theorem even in the first step.
- For $d \geq 3$, there are a lot of resonances.
- We can not use steep lemma directly to remove frequencies.
Elementary regimes
Theorem 4 (Jitomirskaya-L-Shi)

There exist constants $\gamma, \kappa \in (0, 1)$ and large $N_0$ such that if $\log \log \lambda \geq N_0$ and $N \geq N_0$, there is some semi-algebraic set $\Omega_N \subset \mathbb{T}^3$ with $\deg(\Omega_N) \leq N^C$ such that the following holds: $\text{Leb}(\mathbb{T}^3 \setminus \Omega_N) \leq \delta(\lambda)$ ($\delta(\lambda) \to 0$ as $\lambda \to \infty$). For any $\omega \in \Omega_N, E \in \mathbb{R}$, there exists some set $X_N = X_N(\omega, E) \subset \mathbb{T}^3$ such that

\[
\text{Leb}\{\theta \in \mathbb{T} : (x^1, \theta) \in X_N\} \leq e^{-N^\gamma},
\]

\[
\text{Leb}\{\theta \in \mathbb{T}^2 : (\theta, x^2) \in X_N\} \leq e^{-N^\gamma}
\]

and for $x \notin X_N$, the Green’s function $G_N$ is good.
For $d = 1$ and $d = b = 2$, $\Omega_N$ does not depend on $N$. It has full Lebesgue measure, which is given by arithmetic conditions. For example, $\omega$ satisfies Diophantine condition for $d = 1$.

Our proof is inspired by Bourgain ($d = b \geq 3$). We need to use semi-algebraic set (steep lemma) to prove LDT.

The energy elimination is easier than LDT.
Proof of LDT

- Starting scale $N_0$.
- Increase the scale from $N_0$ to $N_1 = e^{N_0 c_1}$ with a hole for all $x$ and $E$: Resolvent identity arguments + multi-scale analysis; Removing frequencies by steep Lemma.
For all $(E, x) \in \mathbb{R} \times \mathbb{T}^3$, there is $N$ satisfying $N_1^{c_4} < N < N_1^{c_5}$ such that if we let

$$U = [-N, N]^2 \setminus [-N^{1/c}, N^{1/c}]^2,$$

then

$$\|G_U(E; x)\| \leq e^{N\kappa},$$

$$\|G_U(E; x)(n, n')\| \leq e^{-\frac{\rho}{2}|n-n'|} \text{ for } |n - n'| > \frac{N}{10}.$$
Except a small set of $x \left( e^{-N_1^\gamma} \right)$, for any $n_0 \in [-N_1, N_1]^2$, there is $N$ (depends on $n_0$) satisfying $N_1^{c_4} < N < N_1^{c_5}$ such that

$$\| G_{E_N}(E; x) \| \leq e^{N\kappa},$$

$$\| G_{E_N}(E; x)(n, n') \| \leq e^{-\frac{\rho}{2}|n-n'|} \text{ for } |n-n'| > \frac{N}{10},$$

where $E_N$ has center $n_0$. 
The Green's function at scale $N_1$ is good based on scale $N$.

Difficulty: the size $N$ changes as the center $n_0$ changes. We need extra work to deal with boundary cases.

- Modify the definition of semi-algebraic set (“good” of Green’s function).
- Modify the “holes”.

\[\text{Diagram showing changes in size and center.}\]
Thank you